Chapter 3. Second Order Linear Equations

Section 3.2. Solutions of Linear Homogeneous Equations; the Wronskian (II)

Recall from part (I)

For

$$L[y] = y'' + p(t)y' + q(t)y = 0,$$

if y_1, y_2 are solutions, then

$$y = c_1 y_1 + c_2 y_2$$

is also a solution.

Wronskian

$$\boxed{W(y_1, y_2)(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}.}$$

If $W(t_0) \neq 0$, then the constants c_1, c_2 can match any initial data at t_0.

When do two solutions give all solutions?

Suppose y_1, y_2 solve $L[y] = 0$.

Theorem 3.2.4

The family

$$\boxed{y = c_1 y_1(t) + c_2 y_2(t)}$$

contains every solution of $L[y] = 0$ if and only if there is at least one point t_0 where

$$\boxed{W(y_1, y_2)(t_0) \neq 0.}$$

Meaning

Two good solutions are enough. But they must be genuinely independent; the Wronskian checks this.

Why Theorem 3.2.4 is true

Let $\phi(t)$ be any solution of $L[y] = 0$.
Assume $W(y_1, y_2)(t_0) \neq 0$.

Record the initial data of ϕ at t_0:

$$y_0 = \phi(t_0), \qquad y_0' = \phi'(t_0).$$

Choose c_1, c_2 so that $c_1 y_1 + c_2 y_2$ has the same two initial data.

Both functions solve the same IVP.

By uniqueness,

$$\boxed{\phi(t) = c_1 y_1(t) + c_2 y_2(t).}$$

Fundamental set of solutions

Definition

Two solutions y_1, y_2 of

$$L[y] = 0$$

form a **fundamental set of solutions** on an interval I if their Wronskian is not zero at some point of I.

Then

The general solution on I is

$$\boxed{y = c_1 y_1 + c_2 y_2.}$$

Thus solving a second order homogeneous linear equation means finding a fundamental set.

Example 1: distinct real exponentials

Suppose

$$y_1(t) = e^{r_1 t}, \qquad y_2(t) = e^{r_2 t}, \qquad r_1 \neq r_2.$$

Then

$$\begin{aligned} W(y_1, y_2)(t) &= \begin{vmatrix} e^{r_1 t} & e^{r_2 t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} \end{vmatrix} \\ &= (r_2 - r_1) e^{(r_1 + r_2)t}. \end{aligned}$$

Conclusion

Since $r_2 - r_1 \neq 0$ and the exponential is never zero,

$$W(t) \neq 0 \quad \text{for all } t.$$

So $e^{r_1 t}$ and $e^{r_2 t}$ form a fundamental set.

Example 2: powers on $t > 0$

Verify that

$$y_1 = t^{1/2}, \qquad y_2 = t^{-1}$$

form a fundamental set for

$$2t^2y'' + 3ty' - y = 0, \qquad t > 0.$$

Wronskian

$$y_1' = \frac{1}{2}t^{-1/2}, \qquad y_2' = -t^{-2}.$$

Thus

$$W = \begin{vmatrix} t^{1/2} & t^{-1} \\ \frac{1}{2}t^{-1/2} & -t^{-2} \end{vmatrix} = -t^{-3/2} - \frac{1}{2}t^{-3/2} = -\frac{3}{2}t^{-3/2}.$$

Since $W \neq 0$ on $t > 0$, these solutions form a fundamental set.

Existence of a fundamental set

Does every equation

$$y'' + p(t)y' + q(t)y = 0$$

have a fundamental set of solutions?

Theorem 3.2.5

Assume p and q are continuous on an interval I, and choose $t_0 \in I$. Let y_1 solve

$$y(t_0) = 1, \qquad y'(t_0) = 0,$$

and let y_2 solve

$$y(t_0) = 0, \qquad y'(t_0) = 1.$$

Then y_1, y_2 form a fundamental set.

Proof idea for Theorem 3.2.5

Existence of y_1 and y_2 follows from the existence theorem.
At $t = t_0$,

$$W(y_1, y_2)(t_0) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1.$$

Conclusion

Since the Wronskian is not zero at t_0, y_1 and y_2 form a fundamental set.

Important distinction

The theorem guarantees that a fundamental set exists; it does not necessarily tell us an easy formula for the two solutions.

Example 3: two fundamental sets for $y'' - y = 0$

A familiar fundamental set is

$$e^t, \qquad e^{-t}.$$

But Theorem 3.2.5 with $t_0 = 0$ asks for solutions satisfying

$$y_1(0) = 1,\ y_1'(0) = 0, \qquad y_2(0) = 0,\ y_2'(0) = 1.$$

These are

$$y_1(t) = \cosh t = \frac{e^t + e^{-t}}{2}, \qquad y_2(t) = \sinh t = \frac{e^t - e^{-t}}{2}.$$

$$W(\cosh t, \sinh t) = \cosh^2 t - \sinh^2 t = 1.$$

Abel's theorem: a formula for the Wronskian

Consider

$$\boxed{y'' + p(t)y' + q(t)y = 0}$$

on an interval where p, q are continuous.

Theorem 3.2.6: Abel's formula

For any two solutions y_1, y_2,

$$\boxed{W(y_1, y_2)(t) = C \exp\left(-\int p(t)\, dt\right),}$$

where C is a constant depending on the pair y_1, y_2.

Consequences

The Wronskian is either identically zero or never zero on the interval.

Proof of Abel's formula

Let

$$W = y_1 y_2' - y_1' y_2.$$

If y_1, y_2 solve

$$y'' + p(t)y' + q(t)y = 0,$$

then

$$y_1'' = -py_1' - qy_1, \qquad y_2'' = -py_2' - qy_2.$$

Differentiate the Wronskian:

$$\begin{aligned} W' &= y_1 y_2'' - y_1'' y_2 \\ &= y_1(-py_2' - qy_2) - (-py_1' - qy_1)y_2 \\ &= -p(y_1 y_2' - y_1' y_2) = -pW. \end{aligned}$$

Thus

$$\boxed{W' + p(t)W = 0.}$$

Solving gives Abel's formula.

Example 4: Abel's formula for the power equation

For

$$2t^2y'' + 3ty' - y = 0, \qquad t > 0,$$

put the equation in standard form:

$$y'' + \frac{3}{2t}y' - \frac{1}{2t^2}y = 0.$$

So

$$p(t) = \frac{3}{2t}.$$

Apply Abel

$$W(t) = C\exp\left(-\int \frac{3}{2t}\,dt\right) = C\exp\left(-\frac{3}{2}\ln t\right) = Ct^{-3/2}.$$

For $y_1 = t^{1/2}$ and $y_2 = t^{-1}$, direct computation gives $C = -3/2$.

Summary

- If two solutions have nonzero Wronskian at one point, they form a fundamental set.
- The general solution is then $y = c_1 y_1 + c_2 y_2$.
- Fundamental sets always exist, although they may not be easy to find explicitly.
- Abel's theorem gives
$$W(t) = C \exp\left(-\int p(t)\,dt\right).$$
- For solutions of the same homogeneous equation, the Wronskian is either always zero or never zero on the interval.