

Chapter 3. Second Order Linear Equations

Section 3.2. Solutions of Linear Homogeneous Equations; the Wronskian (I)

General second order linear equations

A second order linear equation in standard form is

$$y'' + p(t)y' + q(t)y = g(t).$$

Here p, q, g are given functions of t .

Homogeneous

$$y'' + p(t)y' + q(t)y = 0.$$

The right-hand side is zero.

Nonhomogeneous

$$y'' + p(t)y' + q(t)y = g(t).$$

The forcing term $g(t)$ is present.

Operator notation

Let p and q be continuous on an open interval I . Define

$$L[y] = y'' + p(t)y' + q(t)y.$$

Then the homogeneous equation becomes

$$L[y] = 0.$$

Why use L ?

The notation emphasizes that L is a linear operation on functions:

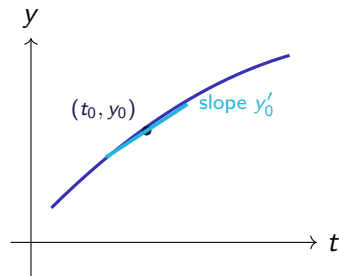
$$L[c_1y_1 + c_2y_2] = c_1L[y_1] + c_2L[y_2].$$

Initial data for a second order equation

For a second order equation, an initial value problem needs two conditions:

$$y(t_0) = y_0, \quad y'(t_0) = y'_0.$$

The first condition fixes the point through which the solution curve passes.
The second condition fixes the slope at that point.



Theorem 3.2.1: existence and uniqueness

Consider

$$y'' + p(t)y' + q(t)y = g(t), \quad y(t_0) = y_0, \quad y'(t_0) = y'_0.$$

Existence and uniqueness theorem

If p, q, g are continuous on an open interval I containing t_0 , then the initial value problem has exactly one solution on the whole interval I .

What the theorem gives

- ▶ Existence: there is a solution.
- ▶ Uniqueness: no second solution has the same initial data.
- ▶ Interval: the solution is guaranteed on the interval where the coefficients are continuous.

How to find the guaranteed interval

Start with a linear equation not necessarily in standard form:

$$P(t)y'' + Q(t)y' + R(t)y = G(t).$$

To apply the theorem, divide by $P(t)$:

$$y'' + \frac{Q(t)}{P(t)}y' + \frac{R(t)}{P(t)}y = \frac{G(t)}{P(t)}.$$

Procedure

1. Find where $P(t) = 0$ or where any coefficient is discontinuous.
2. Choose the largest open interval containing t_0 and avoiding those bad points.
3. On that interval, the theorem guarantees a unique solution.

Example 1: interval of existence

Find the longest interval on which the IVP is guaranteed to have a unique solution:

$$(t^2 - 3t)y'' + ty' - (t + 3)y = 0,$$
$$y(1) = 2, \quad y'(1) = 1.$$

Standard form

Since $t^2 - 3t = t(t - 3)$,

$$y'' + \frac{1}{t-3}y' - \frac{t+3}{t(t-3)}y = 0.$$

The discontinuities occur at

$$t = 0, \quad t = 3.$$

Because $t_0 = 1$, the largest open interval containing 1 is

$$0 < t < 3.$$

Zero initial data gives the zero solution

Consider

$$y'' + p(t)y' + q(t)y = 0, \quad y(t_0) = 0, \quad y'(t_0) = 0.$$

Clearly

$$y(t) \equiv 0$$

is a solution.

Using uniqueness

By Theorem 3.2.1, it is the only solution with these initial conditions.

Important consequence

A nonzero solution of a homogeneous linear equation cannot have both

$$y(t_0) = 0 \quad \text{and} \quad y'(t_0) = 0$$

at the same point.

Superposition: the key homogeneous property

Suppose

$$L[y_1] = 0, \quad L[y_2] = 0.$$

Then, for any constants c_1, c_2 ,

$$L[c_1 y_1 + c_2 y_2] = 0.$$

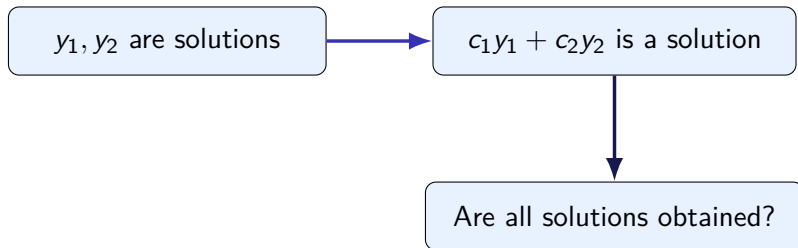
Reason

$$\begin{aligned} L[c_1 y_1 + c_2 y_2] &= c_1 L[y_1] + c_2 L[y_2] \\ &= c_1 \cdot 0 + c_2 \cdot 0 = 0. \end{aligned}$$

This is called the **principle of superposition**.

What superposition does not say

Superposition says that $c_1y_1 + c_2y_2$ is a solution if y_1, y_2 are solutions. It does **not** yet say that every solution has this form.



Next idea

The Wronskian answers this question.

Trying to match initial data

Assume y_1 and y_2 are solutions of $L[y] = 0$. We try

$$y = c_1 y_1 + c_2 y_2.$$

To satisfy

$$y(t_0) = y_0, \quad y'(t_0) = y'_0,$$

we need

$$\begin{cases} c_1 y_1(t_0) + c_2 y_2(t_0) = y_0, \\ c_1 y'_1(t_0) + c_2 y'_2(t_0) = y'_0. \end{cases}$$

Question

When does this 2×2 linear system have a unique solution for c_1, c_2 ?

The determinant: Wronskian

The coefficient determinant is

$$W(y_1, y_2)(t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0).$$

Definition

For each t in the interval,

$$W(y_1, y_2)(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t).$$

This determinant is called the **Wronskian** of y_1 and y_2 .

Why $W(t_0) \neq 0$ matters

If

$$W(y_1, y_2)(t_0) \neq 0,$$

then for every pair of initial data (y_0, y'_0) , there is a unique pair (c_1, c_2) .

Cramer's rule

$$c_1 = \frac{y_0 y'_2(t_0) - y'_0 y_2(t_0)}{W(y_1, y_2)(t_0)},$$

$$c_2 = \frac{-y_0 y'_1(t_0) + y'_0 y_1(t_0)}{W(y_1, y_2)(t_0)}.$$

Then $y = c_1 y_1 + c_2 y_2$ satisfies both the equation and the prescribed initial data.

Example 2: Wronskian of two exponentials

For

$$y'' + 5y' + 6y = 0,$$

we know that

$$y_1 = e^{-2t}, \quad y_2 = e^{-3t}$$

are solutions.

Compute

$$W(y_1, y_2)(t) = \begin{vmatrix} e^{-2t} & e^{-3t} \\ -2e^{-2t} & -3e^{-3t} \end{vmatrix} = -e^{-5t}.$$

Since $e^{-5t} > 0$, we have

$$W(y_1, y_2)(t) \neq 0 \quad \text{for every } t.$$

Theorem 3.2.3: matching initial conditions

Let y_1, y_2 be two solutions of

$$L[y] = y'' + p(t)y' + q(t)y = 0.$$

Wronskian criterion at t_0

The constants c_1, c_2 can be chosen so that

$$y = c_1 y_1 + c_2 y_2$$

satisfies any assigned initial conditions at t_0 if and only if

$$W(y_1, y_2)(t_0) \neq 0.$$

Summary

- ▶ Continuity of p, q, g on an interval gives existence and uniqueness for the IVP.
- ▶ Homogeneous linear equations satisfy superposition.
- ▶ The Wronskian is

$$W(y_1, y_2) = y_1 y_2' - y_1' y_2.$$

- ▶ If $W(y_1, y_2)(t_0) \neq 0$, then $c_1 y_1 + c_2 y_2$ can satisfy any initial data at t_0 .