

Chapter 3. Second Order Linear Equations

Section 3.1. Homogeneous Equations with Constant Coefficients (II)

The initial value problem

Consider

$$ay'' + by' + cy = 0, \quad y(t_0) = y_0, \quad y'(t_0) = v_0.$$

Assume the characteristic equation has real distinct roots r_1, r_2 . Then

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t},$$

$$y' = c_1 r_1 e^{r_1 t} + c_2 r_2 e^{r_2 t}.$$

Apply the initial conditions:

$$c_1 e^{r_1 t_0} + c_2 e^{r_2 t_0} = y_0,$$

$$c_1 r_1 e^{r_1 t_0} + c_2 r_2 e^{r_2 t_0} = v_0.$$

A useful formula for $t_0 = 0$

If the initial conditions are given at $t_0 = 0$, then

$$y(0) = y_0, \quad y'(0) = v_0.$$

The equations for c_1, c_2 are simply

$$c_1 + c_2 = y_0,$$

$$r_1 c_1 + r_2 c_2 = v_0.$$

Solving,

$$c_1 = \frac{v_0 - r_2 y_0}{r_1 - r_2}, \quad c_2 = \frac{r_1 y_0 - v_0}{r_1 - r_2}.$$

Classroom advice

You do not need to memorize this formula. Usually it is safer to set up and solve the two linear equations directly.

Example 1: solve an IVP

Solve

$$y'' + 5y' + 6y = 0, \quad y(0) = 2, \quad y'(0) = 3.$$

The characteristic equation is

$$r^2 + 5r + 6 = 0,$$

so

$$r = -2, -3.$$

Hence

$$y = c_1 e^{-2t} + c_2 e^{-3t}.$$

Differentiate:

$$y' = -2c_1 e^{-2t} - 3c_2 e^{-3t}.$$

Example 1: determine the constants

Use

$$y = c_1 e^{-2t} + c_2 e^{-3t}, \quad y' = -2c_1 e^{-2t} - 3c_2 e^{-3t}.$$

From $y(0) = 2$,

$$c_1 + c_2 = 2.$$

From $y'(0) = 3$,

$$-2c_1 - 3c_2 = 3.$$

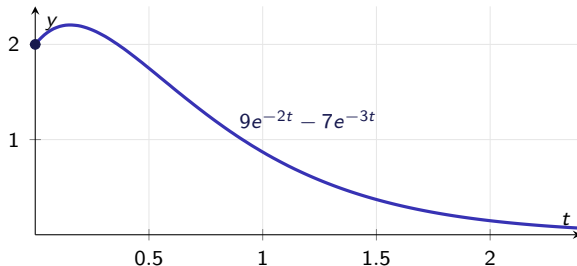
Solving,

$$c_1 = 9, \quad c_2 = -7.$$

Therefore

$$y(t) = 9e^{-2t} - 7e^{-3t}.$$

Example 1: graph and behavior



- ▶ The solution initially increases because $y'(0) = 3 > 0$.
- ▶ Since both exponents are negative, $y(t) \rightarrow 0$ as $t \rightarrow \infty$.

Example 2: another IVP

Solve

$$4y'' - 8y' + 3y = 0, \quad y(0) = 2, \quad y'(0) = \frac{1}{2}.$$

The characteristic equation is

$$4r^2 - 8r + 3 = 0.$$

By the quadratic formula,

$$r = \frac{8 \pm \sqrt{64 - 48}}{8} = \frac{8 \pm 4}{8}.$$

Thus

$$r_1 = \frac{3}{2}, \quad r_2 = \frac{1}{2}.$$

Therefore

$$y = c_1 e^{3t/2} + c_2 e^{t/2}.$$

Example 2: use the initial conditions

We have

$$y = c_1 e^{3t/2} + c_2 e^{t/2}.$$

Then

$$y' = \frac{3}{2}c_1 e^{3t/2} + \frac{1}{2}c_2 e^{t/2}.$$

Use $y(0) = 2$ and $y'(0) = \frac{1}{2}$:

$$\begin{aligned}c_1 + c_2 &= 2, \\ \frac{3}{2}c_1 + \frac{1}{2}c_2 &= \frac{1}{2}.\end{aligned}$$

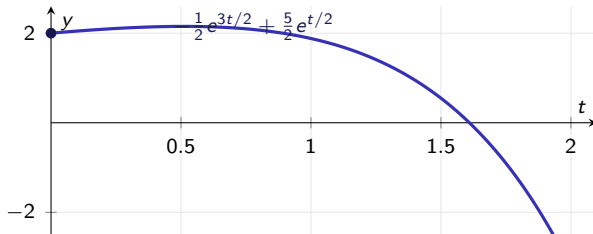
Solving,

$$c_1 = -\frac{1}{2}, \quad c_2 = \frac{5}{2}.$$

Hence

$$y(t) = -\frac{1}{2}e^{3t/2} + \frac{5}{2}e^{t/2}.$$

Example 2: graph and behavior



- ▶ Both exponentials grow as t increases.
- ▶ The term $-\frac{1}{2}e^{3t/2}$ eventually dominates.
- ▶ Thus $y(t) \rightarrow -\infty$ as $t \rightarrow \infty$.

Finding a maximum point

Return to Example 1:

$$y(t) = 9e^{-2t} - 7e^{-3t}.$$

It begins by increasing but tends to zero later. Therefore it should have a maximum.

Calculus idea

At a maximum point with a smooth graph,

$$y'(t) = 0.$$

Differentiate:

$$y'(t) = -18e^{-2t} + 21e^{-3t}.$$

Set equal to zero:

$$-18e^{-2t} + 21e^{-3t} = 0.$$

Maximum point: solve for t

Start from

$$-18e^{-2t} + 21e^{-3t} = 0.$$

Move one term:

$$21e^{-3t} = 18e^{-2t}.$$

Divide by e^{-3t} :

$$21 = 18e^t.$$

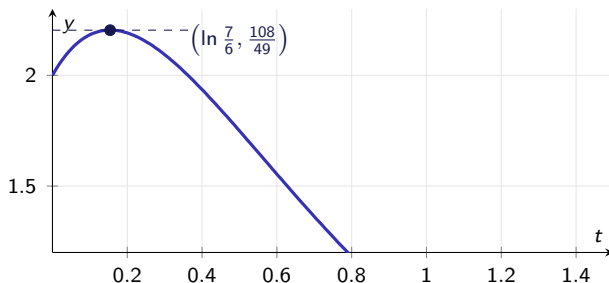
Hence

$$e^t = \frac{7}{6}, \quad \boxed{t_m = \ln \frac{7}{6}}.$$

The maximum value is

$$y(t_m) = 9e^{-2t_m} - 7e^{-3t_m} = \boxed{\frac{108}{49}} \approx 2.204.$$

Maximum point: picture



Comment

Solving the differential equation is not the final goal. We also want to understand the graph and behavior of the solution.

Long-time behavior from roots

For real distinct roots,

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

What happens as $t \rightarrow \infty$?

1. If $r_1 < 0$ and $r_2 < 0$, then $y(t) \rightarrow 0$.
2. If at least one root is positive, some nonzero solutions grow exponentially.
3. If one root is 0 and the other is negative, solutions may approach a constant.

The largest root usually controls the eventual behavior.

Behavior examples

Both negative

$$r_1 = -2, \quad r_2 = -3$$

$$y = c_1 e^{-2t} + c_2 e^{-3t}$$

Typically $y \rightarrow 0$.

One positive

$$r_1 = 1, \quad r_2 = -3$$

$$y = c_1 e^t + c_2 e^{-3t}$$

Usually grows like e^t .

One zero

$$r_1 = 0, \quad r_2 = -5$$

$$y = c_1 + c_2 e^{-5t}$$

Approaches c_1 .

How to sketch solutions

For

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t},$$

a quick sketch can be made using four pieces of information.

Initial value: $y(0)$.

Initial slope: $y'(0)$.

Long-time behavior from the signs of r_1, r_2 .

Turning points from $y'(t) = 0$.

Do not over-plot

For hand sketches, the qualitative behavior is more important than exact numerical accuracy.