Chapter 1. Introduction

Section 1. Basic Mathematical Models; Direction Fields

Warm-up: differential equations and initial data

Differential equation

An equation involving an unknown function and one or more derivatives.

$$y'(t) = t$$

A solution is a function whose derivative is t.

$$y(t) = \frac{1}{2}t^2 + C.$$

Initial value problem

Add one value to choose one curve:

$$y'(t) = t, \qquad y(0) = \frac{1}{2}.$$

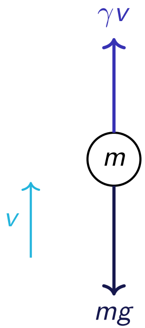
Then $C = \frac{1}{2}$, so

$$\boxed{y(t) = \frac{1}{2}t^2 + \frac{1}{2}.}$$

Example 1: a falling object

Variables and parameters

- t: time in seconds.
- $v(t)$: velocity, positive downward.
- m: mass of the object.
- $g \approx 9.8\,\mathrm{m/s^2}$: gravity.
- γ: drag coefficient.



Choose downward as positive.

Newton's law

$$F = ma = m\frac{dv}{dt}.$$

Deriving the falling object equation

Gravity acts downward; drag acts opposite to the motion. Therefore

$$m\frac{dv}{dt} = mg - \gamma v.$$

Dividing by m gives

$$\boxed{\frac{dv}{dt} = g - \frac{\gamma}{m}v.}$$

Special numerical case

If $m = 10$ kg and $\gamma = 2$ kg/s, then

$$\frac{dv}{dt} = 9.8 - \frac{1}{5}v.$$

Interpretation: the sign of dv/dt tells whether the speed is increasing or decreasing.

Direction fields: read the slope before solving

For an equation

$$\frac{dy}{dt} = f(t, y),$$

at each point (t, y) draw a short segment with slope $f(t, y)$.

Why useful?

- No explicit solution is required.
- Gives a qualitative picture.
- Shows increasing/decreasing behavior.
- Helps locate equilibrium solutions.

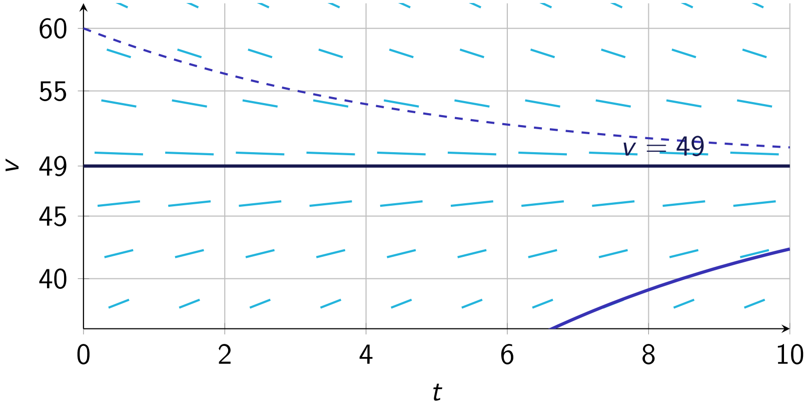
Class prompt

For $v' = 9.8 - v/5$:

$$\left.\frac{dv}{dt}\right|_{v=40} = 1.8 > 0, \qquad \left.\frac{dv}{dt}\right|_{v=50} = -0.2 < 0.$$

What velocity separates these two behaviors?

Direction field for $v' = 9.8 - v/5$



Above 49, slopes are negative; below 49, slopes are positive.

Equilibrium and terminal velocity

Set the derivative equal to zero:

$$0 = 9.8 - \frac{1}{5}v \quad \Longrightarrow \quad \boxed{v = 49.}$$

Interpretation

The constant solution $v(t) = 49$ is an equilibrium solution, also called the terminal velocity in this model.

$$v < 49 \Rightarrow v' > 0$$

The object speeds up.

$$v > 49 \Rightarrow v' < 0$$

The object slows down.

Example 2: field mice and owls

Let $p(t)$ be the field mouse population at time t in months.

Growth without predators

Assume growth rate is proportional to population:

$$\frac{dp}{dt} = rp.$$

For $r = 0.5$ per month,

$$p' = 0.5p.$$

Add owl predation

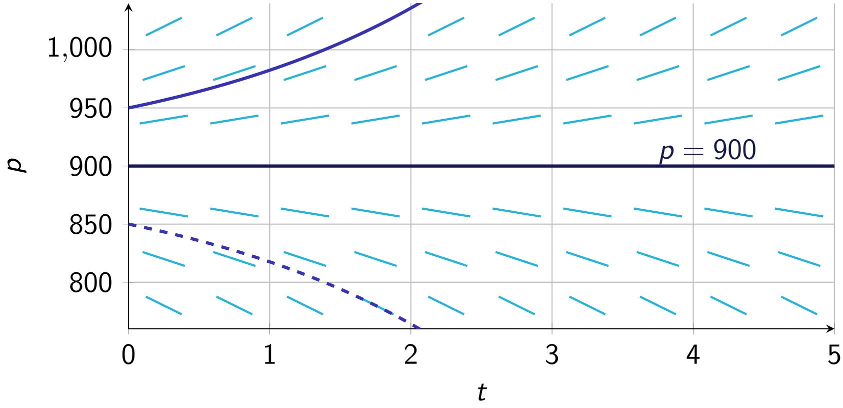
Owls kill 15 mice per day.

$$15 \times 30 = 450 \quad \text{mice/month.}$$

So the model is

$$\boxed{\frac{dp}{dt} = 0.5p - 450.}$$

Direction field for $p' = 0.5p - 450$



$p = 900$ is an equilibrium, but nearby solutions move away from it.

General equation: $y' = ay - b$

Consider

$$\frac{dy}{dt} = ay - b, \qquad y(0) = y_0,$$

where $a \neq 0$ and b are constants.

Equilibrium solution

$$0 = ay - b \quad \Longrightarrow \quad \boxed{y_e = \frac{b}{a}.}$$

Separate and integrate

$$\frac{dy/dt}{y - b/a} = a \quad \Longrightarrow \quad \ln\left|y - \frac{b}{a}\right| = at + C.$$

$$\boxed{y(t) = \frac{b}{a} + ce^{at}.}$$

Solving the initial value problem

Use $y(0) = y_0$:

$$y_0 = \frac{b}{a} + c \quad \Longrightarrow \quad c = y_0 - \frac{b}{a}.$$

Therefore the solution is

$$\boxed{y(t) = \frac{b}{a} + \left(y_0 - \frac{b}{a}\right) e^{at}.}$$

Summary

Key concepts

- A differential equation describes a rate of change.
- An initial condition selects one solution curve.
- Direction fields show qualitative behavior.
- Equilibria occur where the derivative is zero.

Key formula

For

$$y' = ay - b, \qquad y(0) = y_0,$$

with $a \neq 0$,

$$\boxed{y(t) = \frac{b}{a} + \left(y_0 - \frac{b}{a}\right) e^{at}.}$$