

Introduction to SDE Models in Population Biology

Lecture 3: Population Systems—Species Interaction and Epidemics

2-hour PhD short course – 50-page beginner version

Learning objectives and two-hour plan

Students should be able to

- formulate multidimensional population SDEs with biologically appropriate noise;
- prove positive invariance using logarithmic transforms and localization;
- compute rare-species growth rates on boundary regimes;
- analyze stochastic competition and predator–prey thresholds;
- identify how Itô corrections modify epidemic invasion conditions.

Suggested timing: 0–25 min multidimensional SDE tools; 25–50 min positive cone and correlated noise; 50–60 min break; 60–90 min competition; 90–105 min predator–prey; 105–118 min epidemic model; 118–120 min summary.

Textbook guide for this lecture

We build on:

- Mao: multidimensional SDEs, local/global solutions, Lyapunov methods, stochastic stability;
- Øksendal: multidimensional Itô formula, generators, Markov diffusion viewpoint.

The new biological idea is **boundary invasion analysis**: when one species is rare, its logarithmic growth is averaged along the stochastic dynamics of the resident species.

A general stochastic population system

A useful form is

$$dX_i(t) = X_i(t) \left[f_i(X(t))dt + \sum_{j=1}^m \sigma_{ij}(X(t))dW_j(t) \right], \quad i = 1, \dots, n.$$

Here

- $X_i(t)$ is the abundance or density of population i ;
- $f_i(x)$ is its deterministic per-capita growth rate;
- σ_{ij} measures sensitivity to environmental source j ;
- the same Brownian motion may act on several species, creating correlation.

The multiplicative factor X_i makes each coordinate hyperplane $X_i = 0$ dynamically meaningful.

Vector Brownian motion and covariance matrices

Let $W_t = (W_t^1, \dots, W_t^m)^T$ be an m -dimensional Brownian motion with independent components. A general diffusion is

$$dX_t = b(X_t)dt + G(X_t)dW_t,$$

where $G(x)$ is an $n \times m$ matrix. The instantaneous covariance matrix is

$$A(x) = G(x)G(x)^T.$$

Thus

$$dX_t^i dX_t^j = A_{ij}(X_t)dt.$$

Correlated environmental responses can be encoded by sharing Brownian drivers or by choosing a non-diagonal covariance structure.

Why covariance matters biologically

Suppose two species respond to the same environmental driver:

$$dX_t = \cdots + \sigma_1 X_t dW_t,$$

$$dY_t = \cdots + \sigma_2 Y_t dW_t.$$

Then favorable and unfavorable environmental shocks are synchronized. If instead independent Brownian motions are used, short-term shocks are species-specific. The choice changes covariance, invasion rates in more complicated models, and the variability of ratios such as X_t/Y_t .

Multidimensional Itô formula

Let

$$dX = b(X)dt + \Sigma(X)dW, \quad a(X) = \Sigma(X)\Sigma(X)^\top.$$

For $V \in C^2(\mathbb{R}^n)$,

$$dV(X_t) = LV(X_t)dt + \nabla V(X_t)^\top \Sigma(X_t)dW_t,$$

where

$$LV(x) = \sum_i b_i(x) V_{x_i}(x) + \frac{1}{2} \sum_{i,j} a_{ij}(x) V_{x_i x_j}(x).$$

Cross-covariance terms a_{ij} matter when environmental drivers are correlated.

Multidimensional Itô formula in matrix form

For

$$dX_t = b(X_t)dt + G(X_t)dW_t$$

and $f \in C^2(\mathbb{R}^n)$,

$$df(X_t) = \nabla f(X_t)^T b(X_t)dt + \nabla f(X_t)^T G(X_t)dW_t + \frac{1}{2} \text{tr}(A(X_t)D^2 f(X_t)) dt,$$

where

$$A = GG^T.$$

The generator is therefore

$$Lf = \nabla f^T b + \frac{1}{2} \text{tr}(AD^2 f).$$

Worked two-dimensional Itô calculation

Let $f(x, y) = xy$. Then

$$f_x = y, \quad f_y = x, \quad f_{xy} = 1.$$

If

$$dX = b_1 dt + \sigma_1 dW^1, \quad dY = b_2 dt + \sigma_2 dW^2,$$

with

$$dW^1 dW^2 = \rho dt,$$

then

$$d(XY) = Y dX + X dY + \rho \sigma_1 \sigma_2 dt.$$

The covariance term is a genuinely multidimensional Itô correction.

Logarithmic equation for each population

For

$$dX_i = X_i \left[f_i(X) dt + \sum_{j=1}^m \sigma_{ij}(X) dW_j \right],$$

Itô's formula gives, while $X_i > 0$,

$$d \log X_i = \left[f_i(X) - \frac{1}{2} \sum_{j=1}^m \sigma_{ij}^2(X) \right] dt + \sum_{j=1}^m \sigma_{ij}(X) dW_j.$$

Thus the effective per-capita growth rate contains a variance penalty. **Central principle:** extinction and invasion are often decided by long-time averages of this corrected drift.

Componentwise logarithms and per-capita growth

For multiplicative-noise equations

$$dX_i = X_i f_i(X) dt + \sigma_i X_i dW_i,$$

we obtain

$$d \log X_i = \left(f_i(X) - \frac{\sigma_i^2}{2} \right) dt + \sigma_i dW_i.$$

Thus stochastic per-capita growth is obtained by subtracting the Itô correction $\sigma_i^2/2$. On a boundary where species i is absent, averaging this corrected per-capita growth under the resident dynamics leads to the invasion rate.

Positive orthant and invariant faces

If each component has the form

$$dX_i = X_i f_i(X) dt + X_i g_i(X) dW_i,$$

then every coordinate hyperplane $\{X_i = 0\}$ is invariant. Hence the state space contains a hierarchy of invariant faces:

$\{0\}$, single-species axes, two-species faces, $\dots$

These boundary subsystems determine which communities can resist invasion by missing species.

Positive invariance of the orthant

Let

$$\tau_n = \inf \left\{ t \geq 0 : \min_i X_i(t) \leq \frac{1}{n} \text{ or } \max_i X_i(t) \geq n \right\}.$$

On $[0, \tau_n]$, every $\log X_i$ is well defined.

- Apply Itô's formula to $\sum_i [-\log X_i]$ near the boundary.
- Add growth-controlling terms such as $\sum_i X_i$ when needed.
- Derive expectations uniform in n .
- Let $n \rightarrow \infty$ to rule out boundary hitting or explosion.

This is the multidimensional version of the logistic positivity proof.

Independent versus correlated environmental noise

For two species, one may write

$$\begin{aligned}dX &= Xf_1(X, Y)dt + \sigma_1 X dW_1, \\dY &= Yf_2(X, Y)dt + \sigma_2 Y dW_2,\end{aligned}$$

with

$$d\langle W_1, W_2 \rangle_t = \rho dt, \quad -1 \leq \rho \leq 1.$$

- $\rho > 0$: favorable/bad years tend to affect both species in the same direction.
- $\rho < 0$: environmental responses oppose one another.
- Correlation changes joint fluctuations and can affect coexistence statistics even when each single-species log correction remains $-\sigma_i^2/2$.

Two-species stochastic competition

Consider

$$\begin{aligned}dX &= X(a_1 - b_{11}X - b_{12}Y)dt + \sigma_1 X dW_1, \\dY &= Y(a_2 - b_{21}X - b_{22}Y)dt + \sigma_2 Y dW_2.\end{aligned}$$

Assume $b_{ij} > 0$. Important boundary subsystems:

- $(X, Y) = (0, 0)$;
- $Y = 0$: a scalar stochastic logistic equation for X ;
- $X = 0$: a scalar stochastic logistic equation for Y .

The stochastic competition problem is therefore built from the scalar theory of Lecture 2 plus invasion analysis.

Stochastic competition model in full detail

Consider

$$\begin{aligned}dX_t &= X_t(a_1 - b_{11}X_t - b_{12}Y_t)dt + \sigma_1 X_t dW_t^1, \\dY_t &= Y_t(a_2 - b_{21}X_t - b_{22}Y_t)dt + \sigma_2 Y_t dW_t^2.\end{aligned}$$

The corrected intrinsic growth rates are

$$r_i = a_i - \frac{\sigma_i^2}{2}.$$

If $r_i \leq 0$, species i cannot persist even in the absence of its competitor. Thus the first screening step is to determine which single-species boundary systems possess positive invariant laws.

Resident X : complete one-species calculation

On the boundary $Y = 0$,

$$dX_t = X_t(a_1 - b_{11}X_t)dt + \sigma_1 X_t dW_t^1.$$

If $r_1 > 0$, the stationary mean is

$$\bar{X} = \int x \pi_X(dx) = \frac{r_1}{b_{11}}.$$

Near $Y = 0$,

$$d \log Y_t \approx (r_2 - b_{21}X_t) dt + \sigma_2 dW_t^2.$$

Averaging under the resident stationary law gives the invasion rate

$$\lambda_Y = r_2 - b_{21}\bar{X}.$$

Resident X alone: its stationary mean

On the boundary $Y = 0$,

$$dX = X(a_1 - b_{11}X)dt + \sigma_1 X dW_1.$$

If

$$\alpha_1 := a_1 - \frac{\sigma_1^2}{2} > 0,$$

then the X -only process has a stationary distribution with mean

$$\bar{X} = \frac{\alpha_1}{b_{11}}.$$

If $\alpha_1 \leq 0$, species X cannot maintain a persistent one-species resident state. This boundary mean is the quantity that enters the invasion rate of Y .

Invasion rate of Y into the X -resident state

When Y is rare,

$$d \log Y \approx \left(a_2 - b_{21}X_t - \frac{\sigma_2^2}{2} \right) dt + \sigma_2 dW_2.$$

Averaging along the stationary X -resident dynamics suggests

$$\lambda_Y^{(X)} = \left(a_2 - \frac{\sigma_2^2}{2} \right) - b_{21} \frac{a_1 - \sigma_1^2/2}{b_{11}}.$$

- $\lambda_Y^{(X)} > 0$: Y has positive average growth when rare near the X -only regime.
- $\lambda_Y^{(X)} < 0$: the X -resident regime suppresses rare Y on average.

Explicit formula for the invasion rate of Y

Using

$$\bar{X} = \frac{a_1 - \sigma_1^2/2}{b_{11}},$$

we obtain

$$\lambda_Y = a_2 - \frac{\sigma_2^2}{2} - \frac{b_{21}}{b_{11}} \left(a_1 - \frac{\sigma_1^2}{2} \right).$$

This formula separates three effects:

- intrinsic growth of Y ;
- direct environmental-noise penalty on Y ;
- average competitive pressure exerted by resident X .

Four invasion-rate sign patterns

Assume both single-species resident states exist. Then four sign combinations are possible:

$$(\lambda_X, \lambda_Y) \in \{(+, +), (+, -), (-, +), (-, -)\}.$$

Heuristically:

- $(+, +)$ suggests mutual invasibility and possible coexistence;
- $(+, -)$ suggests exclusion of Y by X ;
- $(-, +)$ suggests exclusion of X by Y ;
- $(-, -)$ suggests dependence on initial condition or more complicated boundary selection.

Rigorous conclusions require recurrence, irreducibility, and control of all invariant boundary measures.

The reciprocal invasion rate

Similarly, if

$$\alpha_2 := a_2 - \frac{\sigma_2^2}{2} > 0,$$

then the Y -only resident mean is

$$\bar{Y} = \frac{\alpha_2}{b_{22}},$$

and the invasion rate of X is

$$\lambda_X^{(Y)} = \left(a_1 - \frac{\sigma_1^2}{2} \right) - b_{12} \frac{a_2 - \sigma_2^2/2}{b_{22}}.$$

The signs of the two invasion rates organize the possible long-term scenarios, but a complete theorem also requires recurrence/tightness and exclusion of other boundary behavior.

Comparison with deterministic competition

If $\sigma_1 = \sigma_2 = 0$, then

$$\lambda_Y^{(X)} = a_2 - b_{21} \frac{a_1}{b_{11}}, \quad \lambda_X^{(Y)} = a_1 - b_{12} \frac{a_2}{b_{22}},$$

which are the familiar deterministic invasion expressions. Environmental noise changes these by replacing a_i with

$$a_i - \frac{\sigma_i^2}{2}.$$

Thus even before considering covariance effects, noise can alter whether a resident population persists and whether a missing competitor can invade.

Noise can change competitive outcome

In the deterministic model the resident mean of species X is a_1/b_{11} . Under environmental noise it becomes

$$\frac{a_1 - \sigma_1^2/2}{b_{11}}.$$

Increasing σ_1 weakens the average abundance of resident X , which may increase the invasion rate of Y . At the same time, increasing σ_2 directly lowers Y 's own corrected growth rate. Therefore noise in the resident and noise in the invader can have qualitatively different effects on coexistence.

Ratio method for comparing two species

When both populations remain positive, consider

$$R_t = \log \frac{X_t}{Y_t} = \log X_t - \log Y_t.$$

Then

$$dR_t = [g_X(X_t, Y_t) - g_Y(X_t, Y_t)] dt + \sigma_1 dW_t^1 - \sigma_2 dW_t^2,$$

where g_i denotes corrected per-capita growth. If the drift of R_t has a definite sign on average, the ratio can reveal competitive dominance even when neither equation is explicitly solvable.

A stochastic predator–prey model

Consider

$$\begin{aligned}dX &= X(r - aX - bY)dt + \sigma_1 X dW_1, \\dY &= Y(-d + cX - eY)dt + \sigma_2 Y dW_2.\end{aligned}$$

Here X is prey and Y is predator.

- Without predators, prey follows stochastic logistic growth.
- The predator has negative intrinsic growth $-d$ and relies on prey.
- Predator persistence should depend on its corrected growth rate averaged over the prey-only environment.

Predator–prey model: prey-only boundary dynamics

Consider

$$\begin{aligned}dX_t &= X_t(a - bX_t - cY_t)dt + \sigma_X X_t dW_t^X, \\dY_t &= Y_t(-d + eX_t)dt + \sigma_Y Y_t dW_t^Y.\end{aligned}$$

On $Y = 0$, the prey satisfies a logistic diffusion. If

$$a - \frac{\sigma_X^2}{2} > 0,$$

its stationary mean is

$$\bar{X} = \frac{a - \sigma_X^2/2}{b}.$$

This boundary law determines whether a rare predator can invade.

Deriving the predator invasion rate

For small predator density,

$$d \log Y_t = \left(-d + eX_t - \frac{\sigma_Y^2}{2} \right) dt + \sigma_Y dW_t^Y.$$

Average with respect to the prey-only stationary law:

$$\lambda_Y = -d - \frac{\sigma_Y^2}{2} + e \frac{a - \sigma_X^2/2}{b}.$$

Thus predator persistence requires its average energetic gain from prey to exceed mortality plus the stochastic penalty.

Predator invasion threshold

Assume prey alone is persistent:

$$r - \frac{\sigma_1^2}{2} > 0.$$

Its stationary mean is

$$\bar{X} = \frac{r - \sigma_1^2/2}{a}.$$

For rare predator density,

$$d \log Y \approx \left(-d + cX_t - \frac{\sigma_2^2}{2} \right) dt + \sigma_2 dW_2.$$

Hence the natural predator invasion rate is

$$\lambda_Y = -d - \frac{\sigma_2^2}{2} + c \frac{r - \sigma_1^2/2}{a}.$$

Positive λ_Y indicates average growth of the predator when rare in the prey-only regime.

Environmental correlation in predator–prey systems

If prey and predator are driven by correlated Brownian motions,

$$dW_t^X dW_t^Y = \rho dt,$$

the individual logarithmic drifts are unchanged in this diagonal multiplicative model, but joint fluctuations and quantities such as $\log(X_t/Y_t)$ are affected. Correlation can strongly alter covariance, synchrony, and extinction risk even when first-order invasion formulas look unchanged. This is a reminder that invasion rates do not summarize every aspect of stochastic dynamics.

A Lyapunov candidate for predator–prey recurrence

A common starting point is

$$V(x, y) = c_1 x + c_2 y - \alpha \log x - \beta \log y,$$

with positive constants chosen to balance interaction terms. One computes LV and seeks a drift inequality outside a compact set. The linear terms control large populations, while the logarithms penalize approach to the boundary. Such composite Lyapunov functions are common in rigorous persistence proofs.

A simple stochastic epidemic growth equation

Near a disease-free state, an infected class often has approximately

$$dI_t = I_t(\beta S_t - \gamma)dt + \sigma I_t dW_t.$$

Applying Itô's formula,

$$d \log I_t = \left(\beta S_t - \gamma - \frac{\sigma^2}{2} \right) dt + \sigma dW_t.$$

If the susceptible population is approximately S_0 near disease-free dynamics, the local stochastic invasion quantity becomes

$$\boxed{\beta S_0 - \gamma - \frac{\sigma^2}{2}}.$$

This is the stochastic analogue of linearizing the infected equation at a disease-free state.

From epidemic compartments to an SDE near disease invasion

Suppose I_t is the infected population and the disease-free environment determines a susceptible level near S^* . A linearized stochastic equation may have the form

$$dI_t = I_t(\beta S^* - \gamma)dt + \sigma I_t dW_t.$$

Then

$$d \log I_t = \left(\beta S^* - \gamma - \frac{\sigma^2}{2} \right) dt + \sigma dW_t.$$

Thus the stochastic invasion exponent is

$$\lambda_I = \beta S^* - \gamma - \frac{\sigma^2}{2}.$$

Threshold notation for epidemic models

The sign condition $\lambda_I > 0$ can be rewritten in several ways. For example, when $\gamma + \sigma^2/2 > 0$,

$$\mathcal{R}_s := \frac{\beta S^*}{\gamma + \sigma^2/2} > 1$$

is equivalent to $\lambda_I > 0$. However, the exact stochastic reproduction number depends on the noise model and on which compartment is perturbed. One should derive the threshold from the SDE rather than import a deterministic formula unchanged.

What the epidemic calculation does — and does not — prove

The sign of

$$\beta S_0 - \gamma - \frac{\sigma^2}{2}$$

provides a natural local growth criterion, but a full epidemic theorem may require:

- a coupled SDE system for S , I , and possibly R ;
- proof of positivity and boundedness of the biological region;
- control of the susceptible process near the disease-free boundary;
- a precise definition of extinction, persistence, or stationary endemic behavior.

Research habit: distinguish a heuristic threshold calculation from a theorem with verified hypotheses.

Methods students should recognize in papers

Method	Main use	Typical setting
Comparison	upper/lower scalar processes	competition, epidemics
Log transform	per-capita growth and invasion rates	multiplicative noise
Lyapunov function	non-explosion, tightness, recurrence	nonlinear systems
Invariant measure	average resident environment	boundary subsystems
Boundary growth rate	stochastic persistence/extinction	multispecies systems

Boundary linearization as a general strategy

Across competition, predator–prey, and epidemic examples, the same procedure appears:

- ① identify an invariant boundary subsystem;
- ② determine its long-time law or time averages;
- ③ write the logarithmic equation for a missing species;
- ④ average its corrected per-capita growth along the boundary dynamics;
- ⑤ interpret the resulting invasion rate.

This is the conceptual bridge from elementary SDEs to modern stochastic persistence theory.

Common technical obstacles in multispecies proofs

In research papers, explicit calculations are usually only the beginning. One must also control:

- global existence and positivity;
- tightness of occupation measures;
- invariant probability measures on boundary faces;
- irreducibility or accessibility in the interior;
- martingale terms and uniform moment bounds.

The sign of an invasion rate becomes a rigorous persistence criterion only after these structural properties are established.

In-class model diagnosis

Choose competition, predator–prey, or epidemic dynamics.

- ① Identify the positive state space and all biologically relevant boundary faces.
- ② Write the logarithmic equation for each positive component.
- ③ Select a boundary resident state or invariant regime.
- ④ Compute the corrected rare-species growth rate.
- ⑤ State what extra estimates would be needed to turn the calculation into a theorem.

Deliverable: a 3-minute explanation using one equation and one biological sentence.

Beginner review: vector notation

For n interacting populations, write

$$X_t = (X_1(t), \dots, X_n(t))^T.$$

A general diffusion model can be written as

$$dX_t = b(X_t)dt + \Sigma(X_t)dW_t,$$

where

- $b(x)$ is the drift vector;
- W_t is an m -dimensional Brownian motion;
- $\Sigma(x)$ is an $n \times m$ noise matrix.

The covariance matrix of the noise is

$$a(x) = \Sigma(x)\Sigma^T(x).$$

This matrix determines the second-order part of the generator.

How to simulate correlated environmental noise

Suppose two Brownian motions satisfy

$$dW_1(t)dW_2(t) = \rho dt, \quad -1 \leq \rho \leq 1.$$

A simple construction is

$$W_1(t) = B_1(t), \quad W_2(t) = \rho B_1(t) + \sqrt{1 - \rho^2} B_2(t),$$

where B_1, B_2 are independent Brownian motions. Then

$$dW_1 dW_2 = \rho (dB_1)^2 = \rho dt.$$

Biological interpretation

Positive correlation means both species tend to experience good and bad environmental years together.

Applying Itô to $\log X_i$: the template

Suppose

$$dX_i = X_i f_i(X) dt + X_i \sum_{k=1}^m \sigma_{ik}(X) dW_k(t).$$

Then

$$\frac{dX_i}{X_i} = f_i(X) dt + \sum_k \sigma_{ik}(X) dW_k(t).$$

The quadratic variation term is

$$\frac{(dX_i)^2}{X_i^2} = \sum_{k=1}^m \sigma_{ik}^2(X) dt.$$

Hence

$$d \log X_i = \left(f_i(X) - \frac{1}{2} \sum_{k=1}^m \sigma_{ik}^2(X) \right) dt + \sum_k \sigma_{ik}(X) dW_k(t).$$

This formula is the starting point for invasion-rate calculations.

Boundary faces: deterministic picture first

For two populations, the positive quadrant has boundary faces

$$\{X = 0, Y \geq 0\}, \quad \{X \geq 0, Y = 0\}, \quad \{(0, 0)\}.$$

A boundary face represents a subsystem in which one or more populations are absent. In deterministic population dynamics, invasion analysis asks whether a missing species can increase when rare near a boundary equilibrium. In stochastic population dynamics, the boundary object is often not a single equilibrium but an invariant probability measure for the resident subsystem.

Invasion-rate recipe in four steps

To compute the invasion rate of species Y into a resident X population:

- 1 Set $Y = 0$ and analyze the resident X -subsystem.
- 2 Find its long-time average or invariant probability measure.
- 3 Write the logarithmic equation for Y when Y is rare.
- 4 Average the drift of $\log Y$ along the resident dynamics.

For example, if

$$d \log Y_t = \left(r_2 - b_{21}X_t - b_{22}Y_t - \frac{\sigma_2^2}{2} \right) dt + \sigma_2 dW_2,$$

then near $Y = 0$ the invasion rate is

$$\lambda_Y = r_2 - \frac{\sigma_2^2}{2} - b_{21}\bar{X}.$$

Worked numerical competition example

Assume the X -resident stochastic logistic equation has stationary mean

$$\bar{X} = \frac{r_1 - \sigma_1^2/2}{b_{11}}.$$

Take

$$r_1 = 2, \quad \sigma_1 = 1, \quad b_{11} = 1.$$

Then

$$\bar{X} = 2 - \frac{1}{2} = 1.5.$$

Let species Y have parameters

$$r_2 = 1.4, \quad \sigma_2 = 0.6, \quad b_{21} = 0.7.$$

Then

$$\lambda_Y = 1.4 - \frac{0.6^2}{2} - 0.7(1.5) = 1.4 - 0.18 - 1.05 = 0.17 > 0.$$

So Y can invade the X -resident community.

How environmental correlation enters ratio calculations

Suppose

$$\begin{aligned}d \log X &= \mu_1(X, Y)dt + \sigma_1 dW_1, \\d \log Y &= \mu_2(X, Y)dt + \sigma_2 dW_2,\end{aligned}$$

with

$$dW_1 dW_2 = \rho dt.$$

Then

$$d \log \frac{Y}{X} = (\mu_2(X, Y) - \mu_1(X, Y))dt + \sigma_2 dW_2 - \sigma_1 dW_1.$$

The variance of the noise in the log-ratio is

$$\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2.$$

Positive correlation can reduce fluctuations in relative abundance.

Predator invasion: slow derivation

Consider

$$dY_t = Y_t \left(-d + \frac{mX_t}{h + X_t} - cY_t \right) dt + \sigma_2 Y_t dW_2(t).$$

When the predator is rare, ignore the $-cY_t$ term. Then

$$d \log Y_t = \left(-d + \frac{mX_t}{h + X_t} - \frac{\sigma_2^2}{2} \right) dt + \sigma_2 dW_2(t).$$

If the prey-only process has invariant law π_X , the predator invasion rate is

$$\lambda_Y = \int \left(-d + \frac{mx}{h + x} - \frac{\sigma_2^2}{2} \right) \pi_X(dx).$$

The predator can invade when this average is positive.

Epidemic threshold calculation: what is linearized?

Near the disease-free state, the infected class I is small. A simple linearized stochastic equation has the form

$$dI_t = I_t(\beta S^* - \gamma)dt + \sigma I_t dW_t.$$

Taking logarithms gives

$$d \log I_t = \left(\beta S^* - \gamma - \frac{\sigma^2}{2} \right) dt + \sigma dW_t.$$

Thus the corrected invasion exponent is

$$\lambda_I = \beta S^* - \gamma - \frac{\sigma^2}{2}.$$

Noise reduces the invasion exponent by $\sigma^2/2$ in this elementary multiplicative-noise model.

Common beginner mistakes in multispecies SDEs

- 1 Forgetting the Itô correction in $d \log X_i$.
- 2 Treating correlated Brownian motions as independent.
- 3 Averaging over the wrong resident dynamics.
- 4 Confusing positivity of one invasion rate with full coexistence.
- 5 Ignoring boundary invariant measures with more than one resident species.
- 6 Assuming deterministic thresholds remain unchanged under noise.

A safe habit is to write the logarithmic equation first, identify the resident invariant measure second, and only then average.

Exercises

- Derive both invasion rates for the stochastic competition model.
- Determine how increasing σ_1 changes $\lambda_Y^{(X)}$ when all other parameters are fixed.
- Derive the predator invasion rate from first principles.
- Formulate a stochastic SIS or SIR model and identify its disease-free boundary dynamics.

Reading

- Mao: multidimensional SDEs and Lyapunov methods.
- Øksendal: multidimensional Itô formula and diffusion generators.