

Introduction to SDE Models in Population Biology

Lecture 1: Basic SDE Tools and First Population Models

2-hour PhD short course – 50-page beginner version

Learning objectives and two-hour plan

By the end, students should be able to

- explain how Brownian motion is used to model environmental variability;
- distinguish ordinary, random, and stochastic differential equations;
- use the Itô integral and Itô's formula in elementary calculations;
- solve geometric Brownian motion and interpret its long-term growth rate;
- derive and analyze the logarithmic equation for the stochastic logistic model.

Suggested timing: 0–25 min modeling and Brownian motion; 25–55 min stochastic integral and Itô formula; 55–65 min short break; 65–90 min geometric Brownian motion; 90–115 min stochastic logistic equation; 115–120 min summary.

Textbook guide for this lecture

This lecture follows the standard introductory route in:

- B. Øksendal, *Stochastic Differential Equations: An Introduction with Applications*: Brownian motion, Itô integration, Itô's formula, elementary SDEs;
- X. Mao, *Stochastic Differential Equations and Applications*: strong solutions, basic estimates, and population-style multiplicative-noise examples.

Course emphasis: use only the stochastic-calculus machinery needed to analyze biologically meaningful models. Proofs of measure-theoretic construction are deferred; modeling interpretation and calculations are emphasized.

From deterministic growth to a stochastic model

The deterministic exponential model is

$$\frac{dX}{dt} = aX, \quad X(0) = X_0 > 0.$$

Its per-capita growth rate is constant:

$$\frac{1}{X} \frac{dX}{dt} = a.$$

To model rapid environmental fluctuations, replace the deterministic increment by

$$a dt + \sigma dW_t.$$

This gives

$$dX_t = aX_t dt + \sigma X_t dW_t.$$

Interpretation

a is the average deterministic per-capita growth rate, while σ measures sensitivity to environmental noise. The factor X_t makes the noise proportional to population size.

Three levels of randomness in population models

It is useful to distinguish three modeling levels.

- ① **Deterministic ODE:** parameters and trajectories are fixed once the initial value is given.
- ② **Random ODE:** one or more parameters are random, but after sampling them the equation evolves deterministically.
- ③ **SDE:** randomness acts continuously in time through increments such as dW_t .

For rapidly fluctuating temperature, food availability, or mortality, an SDE often gives the most natural idealization.

Modeling principle

Choose the noise term to represent *how the environment acts on the population*, not merely because a stochastic term is mathematically convenient.

Additive noise versus multiplicative noise

Compare

$$dX_t = f(X_t) dt + \sigma dW_t$$

with

$$dX_t = f(X_t) dt + \sigma X_t dW_t.$$

Additive noise has the same absolute amplitude at all population sizes and may drive a population below zero unless the model is modified near the boundary. **Multiplicative noise** has amplitude proportional to population size; the noise becomes small near $X = 0$ and is common in environmental-noise models. For population biology, positivity considerations often strongly favor multiplicative noise.

Brownian motion: the basic noise process

A standard Brownian motion $W = (W_t)_{t \geq 0}$ satisfies:

- ① $W_0 = 0$;
- ② $W_t - W_s \sim N(0, t - s)$ for $0 \leq s < t$;
- ③ increments over disjoint time intervals are independent;
- ④ sample paths are continuous almost surely.

Consequences:

$$\mathbb{E}[W_t] = 0, \quad \mathbb{E}[W_t^2] = t,$$

and the typical size of an increment over time Δt is $\sqrt{\Delta t}$, not Δt . **Biological message:**

Brownian noise idealizes the accumulated effect of many small, rapidly varying environmental shocks.

Gaussian increments and covariance structure

For $0 \leq s < t$,

$$W_t - W_s \sim N(0, t - s).$$

Thus for small $h > 0$,

$$W_{t+h} - W_t = \sqrt{h} Z, \quad Z \sim N(0, 1).$$

This scaling is the basis of both stochastic calculus and numerical simulation. Moreover,

$$\mathbb{E}[W_s W_t] = \min\{s, t\}.$$

Hence Brownian motion is a centered Gaussian process completely characterized by the covariance kernel $\min\{s, t\}$.

Simulation rule

Over a step of length Δt , generate $\Delta W \approx \sqrt{\Delta t} Z$ with $Z \sim N(0, 1)$.

Quadratic variation: the source of the Itô correction

For a partition $0 = t_0 < \dots < t_n = t$ with mesh tending to zero,

$$\sum_{k=0}^{n-1} (W_{t_{k+1}} - W_{t_k})^2 \longrightarrow t$$

in probability (and along standard refining partitions, almost surely). Thus Brownian motion has nonzero quadratic variation:

$$[W]_t = t.$$

By contrast, a continuously differentiable deterministic function has quadratic variation 0. This is the rigorous content behind the mnemonic

$$(dW_t)^2 = dt.$$

Why ordinary calculus fails

Brownian paths are almost surely nowhere differentiable. Hence an expression such as

$$\frac{dW_t}{dt}$$

is not a classical function. The crucial scaling rule is

$$(\Delta W)^2 \approx \Delta t.$$

In Itô calculus one informally remembers

$$(dW_t)^2 = dt, \quad dW_t dt = 0, \quad (dt)^2 = 0.$$

Why this matters

Terms that would be “second order” in ordinary calculus contribute at first order in time. This creates the Itô correction term and shifts population thresholds.

Filtration and adapted processes: only past information

A filtration $(\mathcal{F}_t)_{t \geq 0}$ represents information available up to time t .

- W_t is $\mathcal{F}_t$ -measurable.
- A process H_t is *adapted* if H_t is known from information available by time t .
- The Itô integral $\int_0^t H_s dW_s$ requires the integrand to be non-anticipating.

For population models, H_t may depend on current population sizes but not on future environmental noise. **Example:** $H_t = X_t$ is adapted when X_t is generated by the SDE up to time t .

The Itô integral for step processes

Suppose

$$H_s = \sum_{k=0}^{n-1} H_{t_k} \mathbf{1}_{[t_k, t_{k+1})}(s), \quad 0 = t_0 < \cdots < t_n = t,$$

with each H_{t_k} known at time t_k . Define

$$\int_0^t H_s dW_s = \sum_{k=0}^{n-1} H_{t_k} (W_{t_{k+1}} - W_{t_k}).$$

The left endpoint is essential: the integrand cannot use the future increment. For square-integrable adapted H ,

$$\mathbb{E} \left[\int_0^t H_s dW_s \right] = 0.$$

Itô isometry: the fundamental estimate

For an adapted square-integrable process H ,

$$\mathbb{E} \left[\left(\int_0^t H_s dW_s \right)^2 \right] = \mathbb{E} \int_0^t H_s^2 ds.$$

This identity does two jobs:

- it defines the integral for general square-integrable adapted processes by approximation;
- it converts a stochastic second moment into an ordinary time integral.

Quick example. If $H_s \equiv c$, then

$$\int_0^t c dW_s = cW_t, \quad \mathbb{E}(cW_t)^2 = c^2 t.$$

A first nontrivial Itô integral

Apply Itô's formula to $f(x) = x^2$ with $X_t = W_t$:

$$d(W_t^2) = 2W_t dW_t + dt.$$

Integrating from 0 to t gives

$$W_t^2 = 2 \int_0^t W_s dW_s + t.$$

Therefore

$$\boxed{\int_0^t W_s dW_s = \frac{1}{2}(W_t^2 - t).}$$

In ordinary calculus one would expect $\frac{1}{2} W_t^2$; the extra $-t/2$ is an Itô correction.

Martingale viewpoint for the Itô integral

If H is adapted and square integrable, then

$$M_t = \int_0^t H_s dW_s$$

is a martingale. Informally,

$$\mathbb{E}[M_t \mid \mathcal{F}_s] = M_s, \quad s < t.$$

Thus the stochastic integral accumulates zero-mean future fluctuations when conditioned on current information. This viewpoint is useful because many long-time SDE proofs have the form

deterministic drift term + martingale term.

The central task is then to show that the martingale term is lower order after division by time.

Itô's formula in one dimension

Let

$$dX_t = b(X_t, t) dt + \sigma(X_t, t) dW_t$$

and let $f \in C^{2,1}$ in (x, t) . Then

$$\begin{aligned} df(X_t, t) &= f_t(X_t, t) dt + f_x(X_t, t) dX_t \\ &\quad + \frac{1}{2} f_{xx}(X_t, t) \sigma^2(X_t, t) dt. \end{aligned}$$

Equivalently,

$$df = \left(f_t + bf_x + \frac{1}{2} \sigma^2 f_{xx} \right) dt + \sigma f_x dW_t.$$

Most useful choices in population biology

$f(x) = \log x$ for per-capita growth; $f(x) = x^p$ for moment estimates; Lyapunov functions for stability and non-explosion.

Itô formula: derivation heuristic

Use a second-order Taylor expansion:

$$\Delta f \approx f_x \Delta X + f_t \Delta t + \frac{1}{2} f_{xx} (\Delta X)^2.$$

For

$$\Delta X \approx b \Delta t + \sigma \Delta W,$$

we have

$$(\Delta X)^2 \approx \sigma^2 (\Delta W)^2 \approx \sigma^2 \Delta t.$$

Terms such as $(\Delta t)^2$ and $\Delta t \Delta W$ are of smaller order. Therefore

$$df = \left(f_t + b f_x + \frac{1}{2} \sigma^2 f_{xx} \right) dt + \sigma f_x dW_t.$$

This heuristic explains exactly where the extra second-derivative term comes from.

Worked calculation: powers of a diffusion

Let

$$dX_t = b(X_t) dt + \sigma(X_t) dW_t,$$

and choose $f(x) = x^p$. Then

$$f'(x) = px^{p-1}, \quad f''(x) = p(p-1)x^{p-2}.$$

Hence

$$\begin{aligned} d(X_t^p) &= pX_t^{p-1}b(X_t) dt + pX_t^{p-1}\sigma(X_t) dW_t \\ &\quad + \frac{1}{2}p(p-1)X_t^{p-2}\sigma^2(X_t) dt. \end{aligned}$$

This identity is the starting point for moment estimates and Lyapunov arguments.

Worked calculation: $f(x) = \log x$

Assume $X_t > 0$ and

$$dX_t = X_t(a - bX_t) dt + \sigma X_t dW_t.$$

Since

$$f'(x) = \frac{1}{x}, \quad f''(x) = -\frac{1}{x^2},$$

Itô's formula gives

$$\begin{aligned} d \log X_t &= \frac{1}{X_t} dX_t - \frac{1}{2} \frac{1}{X_t^2} (\sigma^2 X_t^2) dt \\ &= \left(a - bX_t - \frac{\sigma^2}{2} \right) dt + \sigma dW_t. \end{aligned}$$

Key observation: the effective low-density growth rate is

$$a - \frac{\sigma^2}{2},$$

not simply a .

Example 1: geometric Brownian motion

Consider

$$dX_t = aX_t dt + \sigma X_t dW_t, \quad X_0 > 0.$$

Applying Itô's formula to $Y_t = \log X_t$ gives

$$dY_t = \left(a - \frac{\sigma^2}{2}\right) dt + \sigma dW_t.$$

Integrating,

$$Y_t = \log X_0 + \left(a - \frac{\sigma^2}{2}\right) t + \sigma W_t.$$

Therefore

$$X_t = X_0 \exp \left\{ \left(a - \frac{\sigma^2}{2}\right) t + \sigma W_t \right\}.$$

Positivity is immediate from the explicit formula.

GBM: mean growth versus sample-path growth

From the explicit solution,

$$\mathbb{E}[X_t] = X_0 e^{at},$$

because $\mathbb{E}[e^{\sigma W_t}] = e^{\sigma^2 t/2}$. However,

$$\frac{1}{t} \log X_t = \frac{\log X_0}{t} + a - \frac{\sigma^2}{2} + \sigma \frac{W_t}{t}.$$

Since $W_t/t \rightarrow 0$ almost surely,

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log X_t = a - \frac{\sigma^2}{2} \quad \text{a.s.}$$

Thus $\mathbb{E}[X_t]$ may increase even when a typical sample path decays. This distinction is fundamental in stochastic population dynamics.

Distribution of geometric Brownian motion

From

$$\log X_t = \log X_0 + \left(a - \frac{\sigma^2}{2}\right) t + \sigma W_t,$$

we obtain

$$\log X_t \sim N\left(\log X_0 + \left(a - \frac{\sigma^2}{2}\right) t, \sigma^2 t\right).$$

Thus X_t is lognormally distributed. Its median is

$$X_0 \exp\left(\left(a - \frac{\sigma^2}{2}\right) t\right),$$

while its mean is $X_0 e^{at}$. The difference between mean and median becomes large when $\sigma^2 t$ is large.

GBM threshold from a biological perspective

The almost-sure long-term growth exponent is

$$r_{\text{eff}} = a - \frac{\sigma^2}{2}.$$

Therefore:

- if $a > \sigma^2/2$, a typical trajectory grows exponentially;
- if $a < \sigma^2/2$, a typical trajectory decays exponentially;
- the critical value $a = \sigma^2/2$ separates these two regimes.

Interpretation

Environmental variability reduces the logarithmic growth rate. Even when the arithmetic mean population grows, most sample paths can decline.

Moments of geometric Brownian motion

For $p \in \mathbb{R}$,

$$X_t^p = X_0^p \exp \left\{ p \left(a - \frac{\sigma^2}{2} \right) t + p\sigma W_t \right\}.$$

Hence

$$\mathbb{E}[X_t^p] = X_0^p \exp \left\{ pat + \frac{1}{2}p(p-1)\sigma^2 t \right\}.$$

Alternatively, Itô's formula gives

$$d(X_t^p) = pX_t^p \left(a + \frac{p-1}{2}\sigma^2 \right) dt + p\sigma X_t^p dW_t.$$

Exercise in class: verify the moment formula using this differential equation.

Example 2: stochastic logistic growth

The deterministic logistic equation

$$\dot{X} = X(a - bX), \quad a, b > 0,$$

has carrying capacity $K = a/b$. A common environmental-noise model is

$$dX_t = X_t(a - bX_t) dt + \sigma X_t dW_t.$$

- The drift contains density dependence.
- Multiplicative noise keeps $X = 0$ as a natural boundary.
- At low density the model behaves approximately like geometric Brownian motion.

Therefore the sign of $a - \sigma^2/2$ should be expected to control extinction versus persistence.

Deterministic logistic growth as a benchmark

Before adding noise, consider

$$\dot{x} = x(a - bx), \quad a, b > 0.$$

The equilibria are 0 and

$$K = \frac{a}{b}.$$

For every $x_0 > 0$,

$$x(t) \rightarrow K.$$

The parameter a controls low-density growth and b controls density dependence. The stochastic logistic model

$$dX_t = X_t(a - bX_t)dt + \sigma X_t dW_t$$

will retain density dependence but alter the effective low-density threshold and the long-time distribution.

Logistic model: the corrected low-density growth rate

When X_t is small, the density-dependent term $-bX_t$ is negligible, so

$$d \log X_t \approx \left(a - \frac{\sigma^2}{2} \right) dt + \sigma dW_t.$$

Hence the sign of

$$r = a - \frac{\sigma^2}{2}$$

is expected to determine whether the population can persist away from zero. This simple linearization in logarithmic coordinates foreshadows the invasion-rate method used later for multispecies systems.

A useful explicit transformation for the logistic SDE

Set $Y_t = 1/X_t$. Itô's formula yields the linear SDE

$$dY_t = [b + (\sigma^2 - a)Y_t]dt - \sigma Y_t dW_t.$$

Solving gives the positive representation

$$X_t = \frac{X_0 \exp\{(a - \sigma^2/2)t + \sigma W_t\}}{1 + bX_0 \int_0^t \exp\{(a - \sigma^2/2)s + \sigma W_s\} ds}.$$

Consequence

For $X_0 > 0$, the solution remains positive as long as it exists. The denominator also reveals how density dependence limits growth.

Deriving the reciprocal transformation

For the logistic SDE, let $Y_t = X_t^{-1}$. Then

$$f'(x) = -x^{-2}, \quad f''(x) = 2x^{-3}.$$

Itô's formula gives

$$\begin{aligned} dY_t &= -X_t^{-2} dX_t + X_t^{-3} (\sigma^2 X_t^2) dt \\ &= [b + (\sigma^2 - a) Y_t] dt - \sigma Y_t dW_t. \end{aligned}$$

Thus the nonlinear logistic SDE becomes a linear SDE for $Y_t = 1/X_t$. This transformation is useful for explicit formulas and positivity.

A general linear SDE and integrating factors

Consider

$$dY_t = (\alpha_t Y_t + \beta_t)dt + \gamma_t Y_t dW_t.$$

The homogeneous equation has stochastic exponential

$$\Phi_t = \exp \left\{ \int_0^t \left(\alpha_s - \frac{1}{2} \gamma_s^2 \right) ds + \int_0^t \gamma_s dW_s \right\}.$$

Variation of constants yields

$$Y_t = \Phi_t \left(Y_0 + \int_0^t \Phi_s^{-1} \beta_s ds \right)$$

when the coefficients are deterministic and regular enough. The explicit logistic representation is a special case.

Existence and uniqueness: what we will use

For an SDE

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t,$$

a standard theorem gives a unique local strong solution if b and σ are locally Lipschitz.

- GBM has globally Lipschitz coefficients after factoring X linearly.
- The logistic drift $x(a - bx)$ is locally Lipschitz but has quadratic growth.
- Global existence for biological nonlinearities often needs an additional argument.

In Lecture 2 we will use stopping times and Lyapunov functions to rule out explosion and boundary hitting.

Example 3: Ornstein–Uhlenbeck environmental fluctuations

Not every environmental variable should be modeled by Brownian motion itself. A mean-reverting environmental state can be modeled by

$$dE_t = \kappa(\theta - E_t)dt + \eta dW_t, \quad \kappa > 0.$$

The explicit solution is

$$E_t = \theta + (E_0 - \theta)e^{-\kappa t} + \eta \int_0^t e^{-\kappa(t-s)} dW_s.$$

Hence

$$\mathbb{E}[E_t] \rightarrow \theta, \quad \text{Var}(E_t) \rightarrow \frac{\eta^2}{2\kappa}.$$

This model is useful when the environment has a preferred long-term level rather than independent increments.

Existence and uniqueness: local conditions

For

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t,$$

a standard theorem gives a unique local strong solution if b and σ are locally Lipschitz. If, in addition, a global linear-growth bound holds,

$$|b(x)|^2 + |\sigma(x)|^2 \leq C(1 + |x|^2),$$

then the solution is global. Population models often violate global linear growth because of polynomial density dependence. Lecture 2 will show how Lyapunov functions can replace the linear-growth condition and prevent explosion.

Short workshop: four calculations

Work in pairs for 8–10 minutes.

- 1 For GBM, compute $\mathbb{E}[X_t]$ and $\text{Var}(X_t)$.
- 2 For the logistic SDE, derive $d(X_t^2)$.
- 3 For the logistic SDE, derive $d \log X_t$ and identify the low-density growth rate.
- 4 Explain why additive noise

$$dX_t = X_t(a - bX_t)dt + \sigma dW_t$$

may be biologically problematic near $X = 0$.

Numerical preview: Euler–Maruyama

For

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t,$$

Euler–Maruyama with step Δt is

$$X_{n+1} = X_n + b(X_n)\Delta t + \sigma(X_n)\Delta W_n,$$

where

$$\Delta W_n \sim N(0, \Delta t)$$

independently. For the logistic model,

$$X_{n+1} = X_n + X_n(a - bX_n)\Delta t + \sigma X_n \Delta W_n.$$

Caution: the numerical approximation can become negative even when the exact multiplicative-noise solution stays positive. Positivity-preserving schemes may be preferable in biological applications.

Euler–Maruyama derived from the SDE increment

Over $[t_n, t_{n+1}]$ with step h ,

$$X_{t_{n+1}} - X_{t_n} = \int_{t_n}^{t_{n+1}} b(X_s) ds + \int_{t_n}^{t_{n+1}} \sigma(X_s) dW_s.$$

Freeze the coefficients at the left endpoint:

$$X_{n+1} = X_n + b(X_n)h + \sigma(X_n)\Delta W_n,$$

where

$$\Delta W_n = \sqrt{h} Z_n, \quad Z_n \stackrel{\text{iid}}{\sim} N(0, 1).$$

This is the Euler–Maruyama method.

Numerical caution: positivity may be lost

For multiplicative-noise models, the true solution may remain positive while Euler–Maruyama can produce negative numerical values. For geometric Brownian motion,

$$X_{n+1} = X_n(1 + ah + \sigma\sqrt{h}Z_n),$$

and the factor in parentheses can be negative. Possible remedies include:

- simulating an exact step when available;
- applying Euler–Maruyama to $Y_t = \log X_t$;
- using positivity-preserving or tamed schemes for nonlinear models.

A numerical method should respect biologically essential state constraints whenever possible.

Itô versus Stratonovich: a brief preview

Some applied literature writes stochastic models in Stratonovich form,

$$dX_t = b_S(X_t)dt + \sigma(X_t) \circ dW_t.$$

The equivalent Itô drift is

$$b_I(x) = b_S(x) + \frac{1}{2}\sigma(x)\sigma'(x).$$

For multiplicative noise $\sigma(x) = \alpha x$,

$$b_I(x) = b_S(x) + \frac{1}{2}\alpha^2 x.$$

Thus the interpretation of the stochastic integral changes the drift. In this course we use the Itô convention unless explicitly stated otherwise.

Beginner review: random variables, expectation, variance

Before using Brownian motion, recall the basic probability language.

- A **random variable** Z is a number whose value depends on the outcome ω .
- The **expectation** $\mathbb{E}Z$ is the long-run average value of Z .
- The **variance** is

$$\text{Var}(Z) = \mathbb{E}[(Z - \mathbb{E}Z)^2] = \mathbb{E}Z^2 - (\mathbb{E}Z)^2.$$

- If $Z \sim N(0, 1)$, then $\mathbb{E}Z = 0$ and $\text{Var}(Z) = 1$.

In an SDE, the solution X_t is a random variable for each fixed time t and a random path as t varies.

Normal increments: a concrete calculation

For Brownian motion,

$$W_{t+h} - W_t \sim N(0, h).$$

Equivalently,

$$W_{t+h} - W_t = \sqrt{h} Z, \quad Z \sim N(0, 1).$$

For example, if $h = 0.01$, then a typical increment has size about

$$\sqrt{0.01} = 0.1.$$

This is much larger than $h = 0.01$ itself. Hence stochastic increments are of order $\sqrt{dt}$, not dt .

Key point

The relation $(dW_t)^2 \sim dt$ is the informal source of the extra second-derivative term in Itô's formula.

Informal white noise notation

Many modeling papers write

$$\frac{dX}{dt} = f(X) + g(X)\dot{W}(t).$$

This is only a shorthand. Brownian paths are continuous but nowhere classically differentiable, so $\dot{W}(t)$ is not an ordinary function. The precise SDE notation is

$$dX_t = f(X_t) dt + g(X_t) dW_t.$$

Equivalently,

$$X_t = X_0 + \int_0^t f(X_s) ds + \int_0^t g(X_s) dW_s.$$

Beginner rule

Whenever you see “white noise”, translate it into an integral equation before doing calculations.

How to read the notation $dX_t = b dt + \sigma dW_t$

The differential notation means the following approximate increment over a short time step Δt :

$$\Delta X \approx b \Delta t + \sigma \Delta W, \quad \Delta W \sim N(0, \Delta t).$$

Thus

$$\Delta X \approx b \Delta t + \sigma \sqrt{\Delta t} Z, \quad Z \sim N(0, 1).$$

Interpretation:

- $b\Delta t$ is the deterministic drift contribution;
- $\sigma\sqrt{\Delta t}Z$ is the random fluctuation;
- for small Δt , the random term may be larger than the drift term pathwise.

This explains why SDE paths are rough even when coefficients are smooth.

A simple Brownian simulation recipe

To simulate Brownian motion on $[0, T]$, choose $\Delta t = T/N$ and define

$$W_0 = 0, \quad W_{t_{k+1}} = W_{t_k} + \sqrt{\Delta t} Z_k,$$

where $Z_0, Z_1, \dots$ are independent standard normal random variables. This construction mirrors the defining properties of Brownian motion:

- increments are independent;
- increments are Gaussian;
- the variance of the increment over length Δt is Δt .

Classroom use

Even without proving existence of Brownian motion, this simulation recipe gives a concrete picture of the paths students should have in mind.

The Itô multiplication table

In informal calculations, Itô calculus uses

$$(dt)^2 = 0, \quad dt \, dW_t = 0, \quad (dW_t)^2 = dt.$$

For a process

$$dX_t = b(X_t) \, dt + \sigma(X_t) \, dW_t,$$

we get

$$(dX_t)^2 = \sigma^2(X_t)(dW_t)^2 = \sigma^2(X_t) \, dt.$$

Therefore Taylor's formula gives

$$df(X_t) = f'(X_t)dX_t + \frac{1}{2}f''(X_t)(dX_t)^2,$$

which becomes the one-dimensional Itô formula.

Step-by-step solution of geometric Brownian motion

Consider

$$dX_t = aX_t dt + \sigma X_t dW_t, \quad X_0 > 0.$$

Apply Itô's formula to $f(x) = \log x$:

$$d \log X_t = \frac{1}{X_t} dX_t - \frac{1}{2} \frac{1}{X_t^2} (dX_t)^2.$$

Since $(dX_t)^2 = \sigma^2 X_t^2 dt$,

$$d \log X_t = \left(a - \frac{\sigma^2}{2} \right) dt + \sigma dW_t.$$

Integrating gives

$$\log X_t = \log X_0 + \left(a - \frac{\sigma^2}{2} \right) t + \sigma W_t.$$

Hence

$$X_t = X_0 \exp \left\{ \left(a - \frac{\sigma^2}{2} \right) t + \sigma W_t \right\}.$$

Numerical example: the noise correction matters

Suppose

$$a = 0.08, \quad \sigma = 0.5.$$

The arithmetic mean growth rate is positive:

$$a = 0.08 > 0.$$

But the sample-path growth rate is

$$a - \frac{\sigma^2}{2} = 0.08 - \frac{0.25}{2} = 0.08 - 0.125 = -0.045 < 0.$$

Therefore

$$\frac{1}{t} \log X_t \rightarrow -0.045 \quad \text{a.s.}$$

Biological interpretation

A population can have positive average instantaneous growth but still decline along almost every long sample path when environmental variability is sufficiently strong.

Why logarithms are central in population SDEs

For a positive population process, $\log X_t$ measures cumulative per-capita growth. If

$$dX_t = X_t f(X_t) dt + \sigma X_t dW_t,$$

then Itô's formula gives

$$d \log X_t = \left(f(X_t) - \frac{\sigma^2}{2} \right) dt + \sigma dW_t.$$

Thus the long-time behavior is often governed by the corrected average

$$\text{mean per-capita growth} - \frac{1}{2} \times \text{noise variance}.$$

Method to remember

For multiplicative noise, take logarithms first. This is the most common first step in stochastic population biology.

Extra practice examples for beginners

Use Itô's formula to compute the differential in each case.

- 1 If $dX_t = bX_t dt + \sigma X_t dW_t$, compute $d(X_t^2)$.
- 2 If $dX_t = (aX_t - cX_t^2)dt + \sigma X_t dW_t$, compute $d \log X_t$.
- 3 If $dX_t = -\theta X_t dt + \sigma dW_t$, compute $d(X_t^2)$.
- 4 If $Y_t = e^{-rt} X_t$ and $dX_t = aX_t dt + \sigma X_t dW_t$, compute dY_t .

These are small calculations, but they build the fluency needed to read research papers using stochastic calculus.

Summary and exercises

Main ideas

- Brownian increments scale like $\sqrt{dt}$, producing the Itô correction.
- The logarithmic transform converts multiplicative population noise into additive noise.
- For GBM, the almost-sure growth rate is $a - \sigma^2/2$.
- The same quantity appears at low density in the stochastic logistic model.

Exercises

- Derive $\text{Var}(X_t)$ for GBM.
- Compute $d(X_t^p)$ for stochastic logistic growth.
- Verify the equation for $Y_t = 1/X_t$.
- Simulate GBM for parameters with $a > 0$ but $a - \sigma^2/2 < 0$ and compare sample paths with $\mathbb{E}[X_t]$.

- Øksendal: review the construction/properties of the Itô integral, Itô's formula, and elementary SDE solution methods.
- Mao: review existence and uniqueness of strong solutions and the role of Lyapunov functions in stochastic stability.

Preparation question: Why does local Lipschitz continuity alone not automatically guarantee that a nonlinear population SDE has a global solution for all $t \geq 0$?