Basic Reproduction Numbers for Compartmental Models

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Outline

1. Motivations
2. ODE systems
3. Further developments

A population growth model

Example 1. Consider a single species growth model

$$u' = g(u), \quad u(0) = u_0 \geq 0. \tag{1.1}$$

Assume that $g(0) = 0$. Linearizing (1.1) at $u = 0$, we obtain

$$u' = g'(0)u = bu - au,$$

where b is the intrinsic birth rate, a is the mortality rate. Note that $\frac{1}{a}$ is the average life span. It follows that the basic reproduction number is

$$R_0 = b \cdot 1 \cdot \frac{1}{a} = \frac{b}{a}.$$

Clearly, $u = 0$ is linearly stable if $R_0 < 1$, and linearly unstable if $R_0 > 1$.

An epidemic model

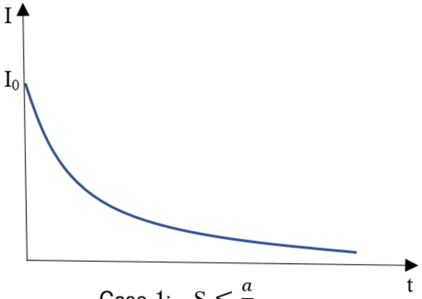
Example 2. For the transmission of a disease, let S be the susceptible, I be the infected, and R be the recovered individuals, respectively. Then we have the following Kermack-McKendrick SIR model (1927):

$$\begin{aligned} \frac{dS}{dt} &= -rSI \\ \frac{dI}{dt} &= rSI - aI \\ \frac{dR}{dt} &= aI \\ S(0) &= S_0 > 0,\ I(0) = I(0) > 0,\ R(0) = 0 \end{aligned} \tag{1.2}$$

where r is the contact rate, and a is the removal rate. Note that $\frac{1}{a}$ is the average infection period. It then follows that system (1.2) admits the following threshold type dynamics.

Threshold dynamics

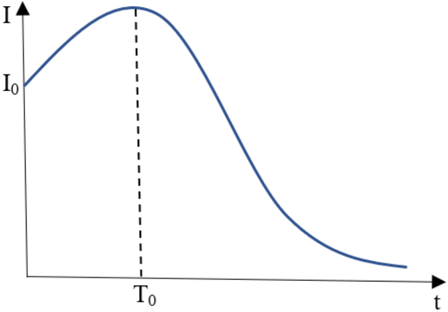
Case 1. If $S_0 \leq \frac{a}{r}$, then both $S(t)$ and $I(t)$ are decreasing in $t \in [0, \infty)$, and $\lim_{t\to\infty} S(t) = S_\infty > 0$ and $\lim_{t\to\infty} I(t) = 0$.



Case 1: $S_0 \leq \frac{a}{r}$

This implies that the disease will decrease to extinction.

Case 2. If $S_0 > \frac{a}{r}$, then there exists $T_0 > 0$ such that $S(T_0) = \frac{a}{r}$, and $S(t)$ is decreasing in $t \in [0, \infty)$ and $\lim_{t\to\infty} S(t) = S_\infty > 0$, while $I(t)$ is increasing in $t \in [0, T_0]$, decreasing in $t \in [T_0, \infty)$, and $\lim_{t\to\infty} I(t) = 0$.



Case 2: $S_0 > \frac{a}{r}$

This implies that the disease will spread for some period of time, and then decrease to extinction.

Basic reproduction number

Initial reproduction number R_0 is the total number of secondary cases (infections) one average infective can produce if the number of susceptibles remained at its initial size S_0. Thus, we have

$$R_0 = rS_0 \cdot 1 \cdot \frac{1}{a} = \frac{rS_0}{a}.$$

K-M Theorem. *If $R_0 \leq 1$, then there is no epidemic outbreak; If $R_0 > 1$, then there is an epidemic outbreak.*

Linearizing (1.2) at the disease-free equilibrium $(S_0, 0)$, we obtain a linear equation for I:

$$\frac{dI}{dt} = rS_0 I - aI := FI - VI = (F - V)I,$$

where $F = rS_0$ is the infection rate, and $V = a$ gives an internal evolution law $\frac{dI}{dt} = -VI$ when there is no infection.

Compartmental models

General approach: O. Diekmann, J. A. P. Heesterbeek and J. A. J. Metz, *J. Math. Biol.*, 28(1990), 365-382.

Compartmental models: P. van den Driessche and J. Watmough, *Math. Biosci.*, 180(2002), 29-48.

Linearizing a given compartmental model at the disease-free equilibrium $(S^*, 0)$, we obtain a liner system for infected variable I:

$$\frac{dI}{dt} = FI - VI = (F - V)I,$$

where F is the infection matrix, and the internal evolution law is $\frac{dI}{dt} = -VI$ when there is no infection. Usually, F is nonnegative, and $-V$ is cooperative. Further, all eigenvalues of $-V$ have negative real parts (**why?**).

Next generation matrix

Let $\psi(0) = (\psi_1(0), \cdots, \psi_m(0))$ be the distribution of infected individuals initially in the infected compartments, and $\psi(t) = (\psi_1(t), \cdots, \psi_m(t))$ be the distribution of these initially infected individuals remaining infected after t time units. Then

$$\frac{d\psi(t)}{dt} = -V\psi(t) \Longrightarrow \psi(t) = e^{-Vt}\psi(0),\ \forall t \geq 0.$$

It follows that the expected distribution of new infectives is

$$\int_0^\infty F \cdot \psi(t)dt = \int_0^\infty F \cdot e^{-Vt}\psi(0)dt = FV^{-1}\psi(0).$$

Then $\psi(0) \longrightarrow FV^{-1}\psi(0)$. Thus, FV^{-1} is called the next generation matrix, and we define its spectral radius as the **basic reproduction number (ratio)**, that is,

$$R_0 = \rho(FV^{-1}).$$

Linear stability

Let $s(A)$ be the **stability modulus** of a matrix A, that is, $s(A) = \max\{Re\lambda : det(\lambda I - A) = 0\}$.

P. van den Driessche and J. Watmough [Math. Biosci., 2002] proved the following result.

Theorem 2.1. *$R_0 - 1$ has the same sign as $s(F - V)$.*

This theorem shows that the sign of $R_0 - 1$ determine the stability of zero solution of $\frac{dI}{dt} = (F - V)I$. Thus, R_0 is a threshold value for the disease invasion.

Applications to various autonomous epidemic models...

An example

Consider a vector-host model for Dengue fever (Feng and Velasco-Hernandez, JMB, 1997):

$$\begin{aligned} \frac{dI}{dt} &= \beta_s SV - (b+\gamma)I, \\ \frac{dV}{dt} &= \beta_m MI - cV, \\ \frac{dS}{dt} &= b - bS + \gamma I - \beta_s SV, \\ \frac{dM}{dt} &= c - cM - \beta_m MI, \end{aligned} \tag{2.1}$$

where I is the number of infected hosts, V is the number of infected vectors, S is the number of susceptible hosts, M is the number of susceptible vectors, β_s and β_m are disease transmission coefficients. The birth rates have been scaled to $b > 0$ for the host and $c > 0$ for the vector.

Basic reproduction number

It is easy to see that $(0,0,1,1)^T$ is the disease-free steady state. Then we have

$$F = \begin{pmatrix} 0 & \beta_s \\ \beta_m & 0 \end{pmatrix}, \qquad V = \begin{pmatrix} b+\gamma & 0 \\ 0 & c \end{pmatrix}.$$

It follows that

$$R_0 = \rho(FV^{-1}).$$

Exercise. Compute the above R_0.

Periodic ODE models

N. Bacaër and S. Guernaoui, *J. Math. Biol.*, 53(2006), 421–436.
W. Wang and X.-Q. Zhao, *J. Dyn. Diff. Equ.*, 20(2008), 699–717.

Consider the linear ω-periodic ODE system (after linearization):

$$\frac{du}{dt} = F(t)u - V(t)u. \tag{3.1}$$

Let $Y(t,s)$ be the evolution matrix of the internal evolution systems $\frac{du}{dt} = -V(t)u$, and let C_ω be the space of continuous ω-periodic (vector-valued) functions.
Wang and Zhao (2008) introduced a linear operator L on C_ω by

$$(L\phi)(t) = \int_0^\infty F(t)Y(t,t-s)\phi(t-s)ds, \ \forall t \in \mathbb{R},$$

and defined $R_0 := r(L)$. It turns out that the sign of $R_0 - 1$ determines the stability of zero solution of system (3.1).

Temporal and spatial heterogeneity

Periodic models (continued):

N. Bacaër and E. H. Ait Dads, *J. Math. Biol.*, 65 (2012), 601-621.

H. Inaba, *J. Math. Biol.*, 22(2012), 113-128.

Impulsive models:

Y. Yang and Y. Xiao, *Nonlinear Analysis (RWA)*, 13(2012), 224-234.

Z. Bai and X.-Q. Zhao, *J. Math. Biol.*, 80(2020), 1095-1117. (Periodic impulsive systems with delay)

Infinite-dimensional population models:

H. Thieme, SIAM J. Appl. Math., 70 (2009), 188-211.

W. Wang and X.-Q. Zhao, *SIAM J. Appl. Dyn. Syst.*, 11(2012),1652–1673.

Temporal and spatial heterogeneity

Almost periodic models:

B.-G. Wang and X.-Q. Zhao, *J. Dyn. Diff. Equ.*, 25(2013), 535–562.

L. Qiang, B.-G. Wang and X.-Q. Zhao, *J. Diff. Equ.*, 269(2020), 4440-4476. (Almost periodic and time delayed systems)

Periodic and time-delayed models:

X.-Q. Zhao, *J. Dyn. Diff. Equ.*, 29(2017), 67-82.

X. Liang, L. Zhang and X.-Q. Zhao, *J. Dyn. Diff. Equ.*, 31(2019), 1247-1278. (Periodic abstract functional differential equations, and a general algorithm for R_0)

Take-home exercise

You may watch 2011 movie "**Contagion**" from the internet.

Thank you!