Uniform Persistence and Coexistence States

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Outline

1. Introduction
2. Uniform Persistence
3. Coexistence States

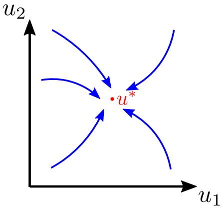
Motivation examples

Example 1. Global stability of a positive equilibrium.

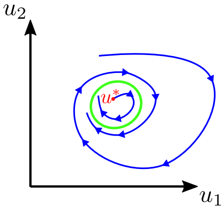
Example 2. Global stability of a positive periodic solution.

Example 3. The interacting species are uniformly persistent.

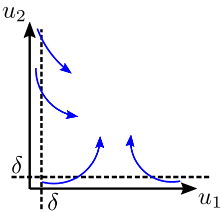
Example 4. The condition that $W^s(M_i) \cap X_0 = \emptyset$ $(1 \leq i \leq k)$ is necessary but not sufficient for uniform persistence.



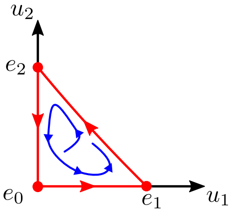
$$\lim_{t\to\infty} u(t) = u^*$$



$$\lim_{t\to\infty} \|u(t) - \overline{u}(t)\| = 0,$$
$$\forall u(0) \in \mathrm{Int}(\mathbb{R}^2_+) \setminus \{u^*\}$$



$$\liminf_{t\to\infty} u_i(t) > \delta, \ \forall i = 1, 2$$



Γ is stable

$$W^s(e_i) \cap X_0 = \emptyset, \ i = 0, 1, 2$$

Abstract persistence

Let X be a complete metric space, and X_0 be an open subset of X. Let $\Phi(t)$ be an autonomous semiflow on X. Define

$$\partial X_0 := X \setminus X_0, \quad M_\partial := \{x \in \partial X_0 : \Phi(t)x \in \partial X_0, \forall t \geq 0\}.$$

Theorem

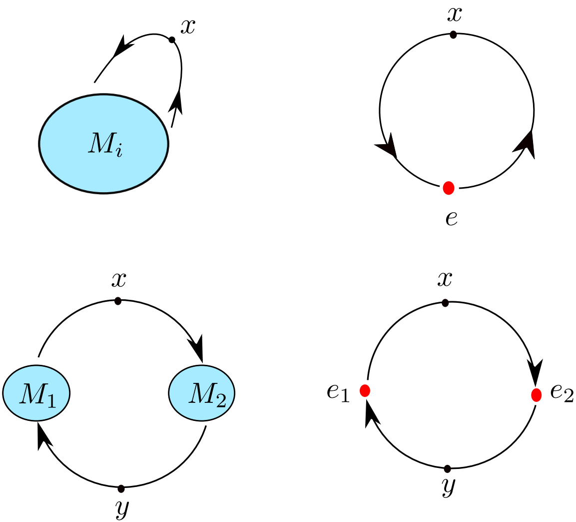
Assume that

(C1) $\Phi(t)(X_0) \subset X_0, \forall t \geq 0$, *and* $\Phi(t)$ *has a global attractor in* X.

(C2) *There exists a finite sequence* $\mathcal{M} = \{M_1, \ldots, M_k\}$ *of disjoint, compact, and isolated invariant sets in* ∂X_0 *such that*

- (a) $\cup_{x \in M_\partial} \omega(x) \subset \cup_{i=1}^{k} M_i$ *(i.e.,*$\forall x \in M_\partial,\ \omega(x) \subset M_i$ *for some* i*);*
- (b) *No subset of* $\mathcal{M}$ *forms a cycle in* ∂X_0*;*
- (c) *Each* M_i *is isolated in* X*;*
- (d) $W^s(M_i) \cap X_0 = \emptyset$ *for each* $1 \leq i \leq k$ *(i.e., no orbit in* X_0 *converges to any* M_i*).*

Then $\exists \delta > 0$ *such that* $\liminf_{t \to \infty} d(\Phi(t)x, \partial X_0) \geq \delta$ *for all* $x \in X_0$.



x
M_i
x
e
x
M_1
M_2
y
x
e_1
e_2
y

Functional differential equations

Let $\Phi(t)$ be the solution semiflow of a scalar FDE model, that is, $\Phi(t)\phi = u_t(\phi)$, $\forall \phi \in X := C([-\tau, 0], \mathbb{R}_+)$. Let $X_0 = X \setminus \{0\}$ and $\partial X_0 = \{0\}$. Assume that $\Phi(t)$ is uniformly persistent with respect to X_0 (i.e., abstract persistence). Then we have

$$\begin{aligned}
\eta &\leq \liminf_{t\to\infty} d(\Phi(t)\phi, \partial X_0) \\
&= \liminf_{t\to\infty} \max_{\theta\in[-\tau,0]} u_t(\phi)(\theta) \\
&= \liminf_{t\to\infty} \max_{\theta\in[-\tau,0]} u(t+\theta, \phi) \\
&= \liminf_{t\to\infty} \max_{s\in[t-\tau,t]} u(s, \phi), \ \forall \phi \in X_0,
\end{aligned}$$

which is different from our desired practical persistence in the sense that there exists $\bar{\eta} > 0$ such that $\liminf_{t\to\infty} u(t, \phi) \geq \bar{\eta}$, $\forall \phi \in X_0$.

Reaction-diffusion equations

Let $\Phi(t)$ be the solution semiflow of a scalar reaction-diffusion model subject to the Robin type boundary condition, that is, $\Phi(t)\phi = u(t, \cdot, \phi)$, $\forall \phi \in X := C(\overline{\Omega}, \mathbb{R}_+)$. Let $X_0 = X \setminus \{0\}$ and $\partial X_0 = \{0\}$. Assume that $\Phi(t)$ is uniformly persistent with respect to X_0 (i.e., abstract persistence). Then we have

$$
\begin{aligned}
\eta &\leq \liminf_{t\to\infty} d(\Phi(t)\phi, \partial X_0) \\
&= \liminf_{t\to\infty} \max_{x\in\overline{\Omega}} u(t, x, \phi),\ \forall \phi \in X_0,
\end{aligned}
$$

which is different from our desired practical persistence in the sense that there exists $\bar{\eta} > 0$ such that $\liminf_{t\to\infty} \min_{x\in\overline{\Omega}} u(t, x) \geq \bar{\eta}$, $\forall \phi \in X_0$.

Generalized distance function

Smith and Zhao, Robust persistence for semidynamical systems, *Nonlinear Analysis, TMA*, 47(2001), 6169–6179.

Definition

A lower semicontinuous function $p : X \to \mathbb{R}_+$ is called a generalized distance function for the semiflow $\Phi(t) : X \to X$ if for every $x \in (X_0 \cap p^{-1}(0)) \cup p^{-1}(0, \infty)$, we have $p(\Phi(t)x) > 0$, $\forall t > 0$.

Note that if $p(x) = d(x, \partial X_0)$, then $p^{-1}(0) = \partial X_0$ and $p^{-1}(0, \infty) = X_0$ **(why?)**.

Recall that $p^{-1}(I) := \{x \in X : p(x) \in I\}$ for any subset I of $\mathbb{R}_+$.

Practical persistence

Smith and Zhao, Robust persistence for semidynamical systems, *Nonlinear Analysis, TMA*, 47(2001), 6169–6179.

Theorem

Let p be a generalized distance function for the semiflow $\Phi(t) : X \to X$. Assume that

(P1) $\Phi(t)$ *has a global attractor in* X;

(P2) *There exists a finite sequence* $M = \{M_1, \ldots, M_k\}$ *of disjoint, compact, and isolated invariant sets in* ∂X_0 *with the following properties:*

- (a) $\cup_{x \in M_\partial} \omega(x) \subset \cup_{i=1}^{k} M_i$;
- (b) *No subset of* M *forms a cycle in* ∂X_0;
- (c) *Each* M_i *is isolated in* X;
- (d) $W^s(M_i) \cap p^{-1}(0, \infty) = \emptyset$ *for each* $1 \leq i \leq k$.

Then $\exists \eta > 0$ *such that* $\liminf_{t \to \infty} p(\Phi(t)x) \geq \eta$ *for all* $x \in X_0$.

Example 1

Let $r \geq 0$ and $C := C([-r, 0], \mathbb{R}^m)$. Consider evolutionary systems of delayed differential equations

$$\begin{aligned} \frac{du(t)}{dt} &= f(u_t), \quad t \geq 0, \\ u_0 &= \phi \in C. \end{aligned} \tag{2.1}$$

Under appropriate assumptions on $f : C \to \mathbb{R}^m$, system (2.1) has a unique solution $u(t, \phi)$ on $[0, \infty)$ for each $\phi \in X := C([-r, 0], \mathbb{R}^m_+)$. Let $\Phi(t)\phi = u_t(\phi)$, and define

$$p(\phi) := \min_{1 \leq i \leq m} \{\phi_i(0)\}, \quad \forall \phi = (\phi_1, \ldots, \phi_m) \in X.$$

Thus, $p : X \to \mathbb{R}_+$ is continuous. In the case where $m = 1$, we have $p(\phi) = \phi(0)$, and hence,

$$p(\Phi(t)\phi) = p(u_t(\phi)) = u_t(\phi)(0) = u(t + 0, \phi) = u(t, \phi).$$

Example 2

Consider reaction–diffusion systems

$$\begin{aligned} \frac{\partial u_i}{\partial t} &= d_i \Delta u_i + f_i(x, u_1, \dots, u_m) \quad \text{in } \Omega \times (0, \infty), \\ B u_i &= 0 \quad \text{on } \partial\Omega \times (0, \infty), \end{aligned} \tag{2.2}$$

where $d_i > 0$, and $Bu = 0$ denotes either Robin type (R) or Dirichlet boundary condition (D). Let $X = C(\overline{\Omega}, \mathbb{R}^m_+)$ in case (R), and $X = C^1_0(\overline{\Omega}, \mathbb{R}^m_+)$ in case (D). Under appropriate assumptions on $f = (f_1, \dots, f_m)$, system (2.2) has a unique solution $u(t, x, \phi)$ on $[0, \infty)$ satisfying $u(0, \cdot, \phi) = \phi$ for each $\phi \in X$, and defines a continuous-time semiflow $\Phi(t)$ on X by $\Phi(t)\phi = u(t, \cdot, \phi)$.

In case (R), we define a continuous function

$$p(\phi) := \min_{1 \le i \le m} \left\{ \min_{x \in \overline{\Omega}} \phi_i(x) \right\}, \quad \forall \phi = (\phi_1, \dots, \phi_m) \in X.$$

For example, letting $m = 1$, we have $p(\phi) = \min_{x\in\overline{\Omega}} \phi(x)$, and hence, $p(\Phi(t)\phi) = p(u(t,\cdot,\phi)) = \min_{x\in\overline{\Omega}} u(t,x,\phi)$. This implies that

$$u(t,x,\phi) \geq p(\Phi(t)\phi), \quad \forall t \geq 0,\, x \in \overline{\Omega}.$$

In case (D), we choose $e \in Int(C_0^1(\overline{\Omega}, \mathbb{R}_+^m))$ and define

$$p(\phi) := \sup\{\beta \in \mathbb{R}_+ : \phi(x) \geq \beta e(x),\, \forall x \in \overline{\Omega},\},$$
$$\forall \phi = (\phi_1, \ldots, \phi_m) \in X.$$

It follows that the above p-function is lower semi-continuous (**why?**). Clearly, $\phi(x) \geq p(\phi)e(x)$, $\forall x \in \overline{\Omega}$, $\phi \in X$. Thus,

$$u(t,x,\phi) = [\Phi(t)\phi](x) \geq p(\Phi(t)\phi)e(x), \quad \forall t \geq 0,\, x \in \overline{\Omega}.$$

Note that $\int_\Omega u_i(t,x,\phi)dx \geq p(\Phi(t)\phi)\int_\Omega e_i(x)dx$, $\forall 1 \leq i \leq m$.

Existence of coexistence states

Let $\Phi(t) : X \to X, t \geq 0$, be a continuous-time semiflow.

Definition 3.1 $x_0 \in X_0$ is called a coexistence state if $\Phi(t)x_0 = x_0,\ t \geq 0$.

Theorem 3.1 (Zhao, CAMQ, 1995) Assume that

(1) $\Phi(t)$ is point dissipative on X;

(2) $\Phi(t)$ is compact for each $t > 0$;

(3) $\Phi(t)$ is uniformly persistent.

Then $\Phi(t)$ has a coexistence state in X_0.

Note that Hutson(1990) and Hofbauer(1990) proved a similar result for autonomous ODE (finite dimension). For more general results under the weak compactness assumption, see Magal and Zhao (SIMA, 2005) (also Zhao's 2017 book).

Existence of coexistence states

Let $S^n : X \to X, n \geq 0$, be a discrete-time semiflow.

Definition 3.2 $x_0 \in X_0$ is called a coexistence state if $S(x_0) = x_0$.

Theorem 3.2 (Zhao, CAMQ, 1995) Assume that

(1) S is point dissipative on X;

(2) S is compact;

(3) S is uniformly persistent.

Then S has a coexistence state in X_0.

Note that if S is the Poincaré map of a periodic system, then the solution with the coexistence state of S as its initial value is a periodic solution in X_0. (e.g., positive periodic solution)

A dynamical approach to some static problems

For example, the positive solutions of an elliptic equation

$$\begin{aligned} d\Delta u + f(u) &= 0, \ x \in \Omega \\ Bu &= 0, \ x \in \partial\Omega \end{aligned}$$

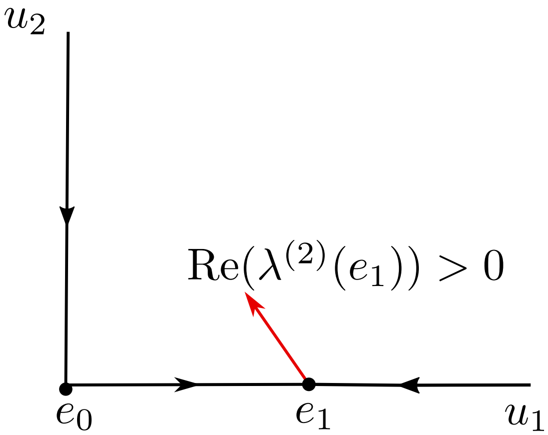
are clearly the steady state solutions of the following parabolic equation

$$\begin{aligned} \frac{\partial u}{\partial t} &= d\Delta u + f(u), \, x \in \Omega, \, t > 0 \\ Bu &= 0, \, x \in \partial\Omega, \, t > 0. \end{aligned}$$

By Theorem 3.1, the uniform persistence of the solution semiflow generated by the parabolic equation implies the existence of a positive solution of the elliptic equation.

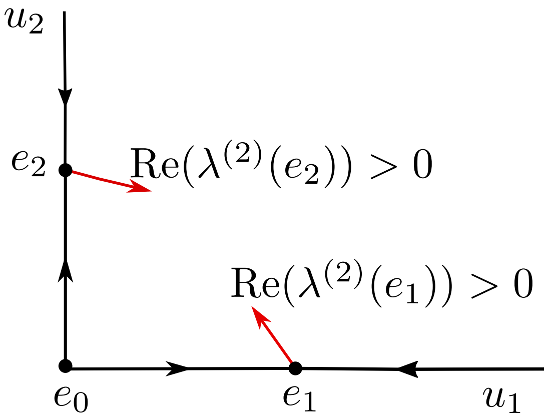
Three application examples

Example 1. The predator-prey population model:



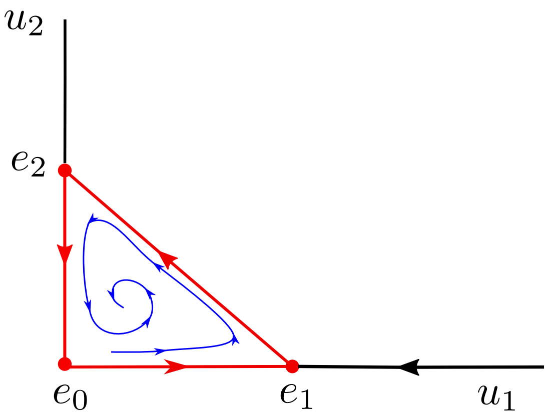
$$M = \{e_0, e_1\}$$

Example 2. Two species competition model:



$$M = \{e_0, e_1, e_2\}$$

Example 3. An ODE model with an unstable heteroclinic orbit:



$M = \{\Gamma\}$, Γ is unstable

Thank you!