Chain Transitive Sets and Limiting Systems

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Outline

Limiting systems

Example 1.1 Let D be a closed subset of $\mathbb{R}^n$. We consider the nonautonomous ordinary differential system:

$$\begin{aligned} &\frac{dx}{dt} = f(t,x),\ t \geq 0, \\ &x(0) = x_0 \in D. \end{aligned} \tag{1.1}$$

Assume that $\lim_{t\to\infty} f(t,x) = f_0(x)$ uniformly for x in any bounded subset of D. Then we have a limiting autonomous system:

$$\begin{aligned} &\frac{dx}{dt} = f_0(x),\ t \geq 0, \\ &x(0) = x_0 \in D. \end{aligned} \tag{1.2}$$

Problem: Under what conditions can we lift the long-time properties of solutions of the limiting system (1.2) to the nonautonomous system (1.1)?

A counterexample

Consider the following planar system (Thieme [JMB, 1992]):

$$\begin{aligned} \frac{dr}{dt} &= r(1-r) \\ \frac{d\theta}{dt} &= \beta r|\sin\theta| + \alpha e^{-\gamma t}, \end{aligned} \tag{1.3}$$

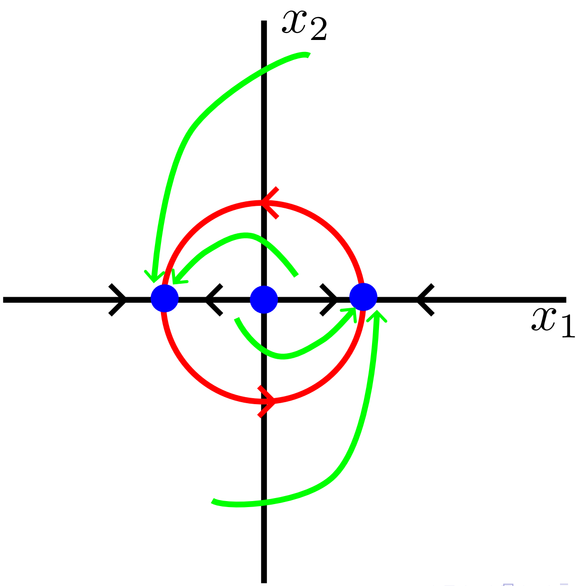
where $\alpha > 0, \beta > \gamma > 0$, $x_1 = r\cos\theta$, and $x_2 = r\sin\theta$. Clearly, system (1.3) has a limiting autonomous system:

$$\begin{aligned} \frac{dr}{dt} &= r(1-r) \\ \frac{d\theta}{dt} &= \beta r|\sin\theta|. \end{aligned} \tag{1.4}$$

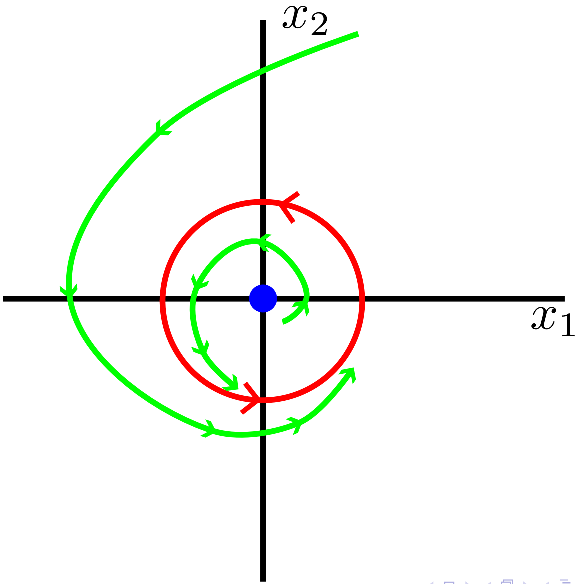
It is easy to see that any forward orbit of system (1.4) converges to one of three equilibria $(0,0)$, $(1,0)$ and $(-1,0)$ as $t \to \infty$.

Exercise 1.1 Show that any nontrivial solution of system (1.3) converges to the circle $r = 1$ as $t \to \infty$.

Phase portrait of autonomous system (1.4)



Dynamics of nonautonomous system (1.3)



Asymptotically autonomous semiflows

To solve the above problem, one may use the theory of asymptotically autonomous semiflows:

H. R. Thieme, Convergence results and Poincaré–Bendixson trichotomy for asymptotically autonomous differential equations, *J. Math. Biology*, 30(1992), 755–763.

K. Mischaikow, H. L. Smith and H. R. Thieme, Asymptotically autonomous semiflows: chain recurrence and Liapunov functions, *Trans. Amer. Math. Soc.*, 347(1995), 1669–1685.

Note that in this theory, the domain of the asymptotically autonomous system is assumed to be the same as that of the limiting autonomous system.

A chemostat model

Example 1.2 Consider the single species growth model in a chemostat:

$$\begin{aligned} \frac{dS}{dt} &= D(S^0 - S) - xP(S) \\ \frac{dx}{dt} &= xP(S) - Dx \\ (S(0), x(0)) &= (S_0, x_0) \in \mathbb{R}^2_+. \end{aligned} \tag{1.5}$$

Here D is the dilution (or washout) rate, S^0 is the inout nutrient concentration, $P(S)$ is the per capita nutrient uptake function. In particular, we take $P(S) = \frac{mS}{a+S}$, where m is maximal growth rate, and a is the Michaelis-Menten (or half-saturation) constant. Both a and m can be measured experimentally.

It is easy to see that $\mathbb{R}^2_+$ is positively invariant for system (1.5).

Let $\Sigma = S + x$. Then system (1.5) is equivalent to the following one:

An equivalent model

$$\begin{aligned} \frac{d\Sigma}{dt} &= DS^0 - D\Sigma \\ \frac{dx}{dt} &= xP(\Sigma - x) - Dx \\ (\Sigma(0), x(0)) &= (\Sigma_0, x_0) \in \Omega := \{(\Sigma, x) : \Sigma \geq x \geq 0\}. \end{aligned} \tag{1.6}$$

It then follows that Ω is positively invariant for system (1.6). Clearly, $x(t)$ satisfies the following nonautonomous equation:

$$\frac{dx}{dt} = xP(\Sigma(t) - x) - Dx. \tag{1.7}$$

Let $\Omega(t) := [0, \Sigma(t)], \forall t \geq 0$. It is easy to see that for any initial value $x(0) \in \Omega(0)$, system (1.7) has a unique solution $x(t)$ such that $x(t) \in \Omega(t), \forall t \geq 0$. Since $\lim_{t \to \infty} \Sigma(t) = S^0$, we have the following limiting system:

Omega limit sets

$$\frac{dx}{dt} = xP(S^0 - x) - Dx. \tag{1.8}$$

Let $\omega = \omega(\Sigma_0, x_0)$ be the omega limit set of the orbit $(\Sigma(t), x(t))$ through (Σ_0, x_0) for the solution semiflow Φ_t of system (1.6), that is,

$$\omega(\Sigma_0, x_0) := \left\{(\Sigma, x) \in \Omega : \exists t_n \to \infty \ s.t. \lim_{n\to\infty} (\Sigma(t_n), x(t_n)) = (\Sigma, x)\right\}.$$

Since $(\Sigma(t), x(t)) \in \Omega$, $\forall t \geq 0$, we have $\Sigma(t) \geq x(t) \geq 0$, and hence, $\omega = \{S^0\} \times \tilde{\omega}$ with $\tilde{\omega} \subset [0, S^0]$.

Let Q_t be the solution semiflow of (1.8) on $[0, S^0]$. Since $\Phi_t(\omega) = \omega$ for all $t \geq 0$, we see that $\Phi_t(S^0, \bar{x}) = (S^0, Q_t(\bar{x}))$, $\forall (S^0, \bar{x}) \in \omega$, $t \geq 0$. It then follows that $Q_t(\tilde{\omega}) = \tilde{\omega}$, $\forall t \geq 0$. But $\tilde{\omega}$ may not be the omega limit set of a forward orbit of (1.8) in $[0, S^0]$.

Note that the nonautonomous system (1.7) has a time-dependent domain $\Omega(t)$, while its limiting system (1.8) has the domain $[0, S^0]$. Thus, we cannot directly use the theory of asymptotically autonomous systems.

Definition

Let (X, d) be a metric space with metric d, and $\Phi(t) : X \to X$, $t \geq 0$, be a continuous-time semiflow.

Definition 2.1 *A nonempty invariant set $A \subset X$ for the semiflow $\Phi(t)$ (that is, $\Phi(t)A = A$, $\forall t \geq 0$) is said to be internally chain transitive if for any $a, b \in A$ and any $\epsilon > 0, t_0 > 0$, there is a finite sequence*

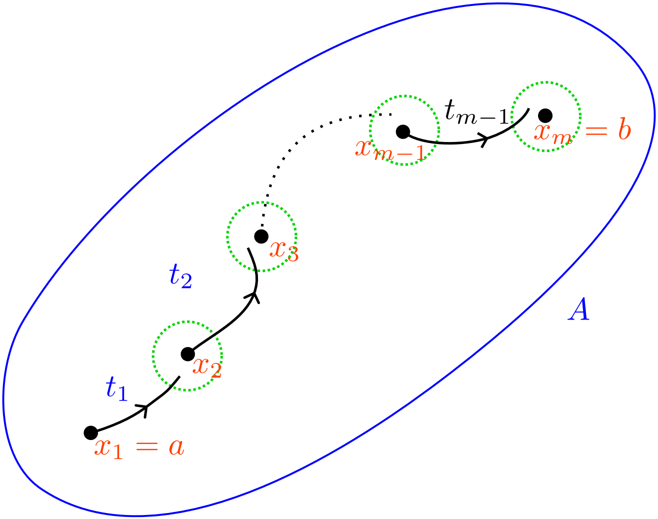
$$\{x_1 = a, x_2, \ldots, x_{m-1}, x_m = b;\ t_1, \ldots, t_{m-1}\}$$

with $x_i \in A$ and $t_i \geq t_0, 1 \leq i \leq m-1$, such that

$$d(\Phi(t_i, x_i), x_{i+1}) < \epsilon, \quad \forall 1 \leq i \leq m-1.$$

The sequence $\{x_1, \ldots, x_m;\ t_1, \ldots, t_{m-1}\}$ is called an (ϵ, t_0)-chain in A connecting a and b.

Chain transitivity



An example for CTS

Lemma 2.1 *Let $\Phi(t) : X \to X$, $t \geq 0$, be a continuous-time semiflow. Then the omega (alpha) limit set of any precompact positive (negative) orbit is internally chain transitive.*

Question: Can a heteroclinic orbit be an omega limit set of a precompact positive orbit?

Exercise 2.1 Show that the set $\tilde{\omega}$ in Example 1.2 is a chain transitive set for the solution semiflow of the limiting system (1.8) on $[0, S^0]$.

Strong attractivity and convergence

Recall that the stable set of an invariant set A for a semiflow $Q(t)$ is defined as

$$W^s(A) := \left\{x \in X : \lim_{t\to\infty} d(Q(t)x, A) = 0\right\}.$$

Theorem 2.1 (Hirsch, Smith and Zhao, 2001, JDDE) *Let A be an attractor and C a compact internally chain transitive set for the autonomous semiflow $Q(t) : X \to X$. If $C \cap W^s(A) \neq \emptyset$, then $C \subset A$.*

Theorem 2.2 (Hirsch, Smith and Zhao, 2001, JDDE) *Assume that each equilibrium of the autonomous semiflow $Q(t) : X \to X$ is an isolated invariant set, that there is no cyclic chain of equilibria, and that every precompact orbit converges to some equilibrium of $Q(t)$. Then any compact internally chain transitive set is an equilibrium of $Q(t)$.*

Discrete-time semiflows

Definition 2.2 *Let (X, d) be a metric space with metric d, and $M : X \to X$ be a continuous map. A nonempty invariant set $A \subset X$ for M (i.e., $M(A) = A$) is said to be internally chain transitive if for any $a, b \in A$ and any $\epsilon > 0$, there is a finite sequence $\{x_1 = a, x_2, \dots, x_{m-1}, x_m = b\}$ with $x_i \in A$ such that*

$$d(M(x_i), x_{i+1}) < \epsilon, \quad \forall 1 \leq i \leq m-1.$$

The sequence $\{x_1, \dots, x_m\}$ is called an ϵ-chain in A connecting a and b.

Remark 2.1 (Hirsch, Smith and Zhao, 2001, JDDE) *Lemma 2.1, Theorems 2.1 and 2.2 are also valid for the discrete-time semiflow $\{M^n\}_{n\geq 0}$.*

Note that $M^n x$, $n \geq 0$ corresponds to $Q(t)x$, $t \geq 0$.

The single species growth model

Now we return to model (1.5) and its equivalent system (1.6).

It is easy to obtain the global dynamics of system (1.8) on $[0, S^0]$.

Lemma 3.1 Assume that $P'(s) > 0, \forall s \geq 0$. Then the following statements are valid:

(a) If $P(S^0) \leq D$, then $x = 0$ is globally asymptotically stable for system (1.8) in $[0, S^0]$.

(b) If $P(S^0) > D$, then system (1.8) has a positive equilibrium $x^* \in (0, S^0)$ and $x = x^*$ is globally asymptotically stable for system (1.8) in $(0, S^0]$.

The global dynamics for model (1.5)

For the chemostat model (1.5), we have the following threshold type result.

Theorem 3.1 Assume that $P'(s) > 0$, $\forall s \geq 0$, and let $(S(t), x(t))$ be the solution of system (1.5). Then the following statements are valid:

(i) If $P(S^0) \leq D$, then $\lim_{t\to\infty}(S(t), x(t)) = (S^0, 0)$ for all $S(0) \geq 0$ and $x(0) \geq 0$.

(ii) If $P(S^0) > D$, then there exists $x^* \in (0, S^0)$ with $P(S^0 - x^*) = D$ such that $\lim_{t\to\infty}(S(t), x(t)) = (S^0 - x^*, x^*)$ for all $S(0) \geq 0$ and $x(0) > 0$.

Proof of Theorem 3.1

Let $\Phi(t)$ be the solution semiflow of system (1.6) on Ω, and $Q(t)$ be the solution semiflow of system (1.8) on $[0, S^0]$. Let ω and $\tilde{\omega}$ be defined as in Example 1.2. By Exercise 2.1, $\tilde{\omega}$ is an internally chain transitive set for $Q(t)$ on $[0, S^0]$.

In the case where $P(S^0) \leq D$, we see from Lemma 3.1 that $W^s(0) = [0, S^0]$, and hence, $\tilde{\omega} \cap W^s(0) \neq \emptyset$. By Theorem 2.1, it then follows that $\tilde{\omega} = \{0\}$, and hence, $\omega = (S^0, 0)$. This implies that $\lim_{t\to\infty}(\Sigma(t), x(t)) = (S^0, 0)$, and hence, $\lim_{t\to\infty} S(t) = \lim_{t\to\infty}(\Sigma(t) - x(t)) = S^0$.

In the case where $P(S^0) > D$, we see from Lemma 3.1 that $W^s(x^*) = (0, S^0]$. Since $x(0) > 0$, we have $x(t) > 0$, $\forall t \geq 0$ (**why?**). Now we show that $\tilde{\omega} \cap W^s(x^*) \neq \emptyset$. Assume, by contradiction, that $\tilde{\omega} \cap W^s(x^*) = \emptyset$. Then $\tilde{\omega} = \{0\}$, and hence, $\omega = (S^0, 0)$. Thus, we have $\lim_{t\to\infty}(\Sigma(t), x(t)) = (S^0, 0)$.

Since $\lim_{t\to\infty} (P(\Sigma(t) - x(t)) - D) = P(S^0) - D > 0$, there exists $T > 0$ such that

$$P(\Sigma(t) - x(t)) - D > \frac{1}{2}\left(P(S^0) - D\right) > 0, \quad \forall t \geq T.$$

Then we have

$$x'(t) \geq x(t) \cdot \frac{1}{2}\left(P(S^0) - D\right), \quad \forall t \geq T.$$

This implies that $x(t) \to \infty$ as $t \to \infty$, a contradiction. Thus, $\tilde{\omega} \cap W^s(x^*) \neq \emptyset$. By Theorem 2.1, it follows that $\tilde{\omega} = x^*$, and hence, $\omega = (S^0, x^*)$. Thus, we have $\lim_{t\to\infty} (\Sigma(t), x(t)) = (S^0, x^*)$, and $\lim_{t\to\infty} S(t) = \lim_{t\to\infty} (\Sigma(t) - x(t)) = S^0 - x^*$.

Exercise 3.1 Use Theorem 2.2 to prove the conclusion (ii) in Theorem 3.1.

References

- M. W. Hirsch, H. L. Smith and Zhao, Chain transitivity, attractivity and strong repellors for semidynamical systems, *J. Dynamics and Differential Equations*, 13(2001), 107–131.
- Zhao, *Dynamical Systems in Population Biology*, second edition, Springer, New York, 2017.