MATH 3161, Midterm Exam, Winter 2026

Name: Student No.:

Instruction: This exam includes 5 questions. All answers must be carefully justified. The value of each problem is marked on the left margin.

[6 Points] 1. Find the general solution of the differential equation $y'' + y = 0$ by means of a power series about $x_0 = 0$.

Solution: Let $y = \sum_{n=0}^{\infty} a_n \cdot x^n$. Then $y' = \sum_{n=1}^{\infty} n \cdot a_n x^{n-1}$, and

$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$. Thus, $\sum_{n=2}^{\infty} n(n-1) a_n \cdot x^{n-2} + \sum_{n=0}^{\infty} a_n \cdot x^n = 0$,

and hence, $\sum_{n=0}^{\infty} \left[(n+2)(n+1) \cdot a_{n+2} + a_n \right] x^n = 0$

$\Rightarrow (n+2)(n+1) a_{n+2} + a_n = 0, \quad \forall n \geq 0$

$\Rightarrow a_{n+2} = -\frac{a_n}{(n+2)\cdot(n+1)}, \quad \forall n \geq 0$

$\Rightarrow a_{2k} = \frac{(-1)^k \cdot a_0}{(2k)!}, \quad \forall k \geq 0,$

and $a_{2k+1} = \frac{(-1)^k \cdot a_1}{(2k+1)!}, \quad \forall k \geq 0.$

Thus, $y = \sum_{k=0}^{\infty} a_{2k} \cdot x^{2k} + \sum_{k=0}^{\infty} a_{2k+1} \cdot x^{2k+1}$

$= a_0 \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} + a_1 \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}.$

[6 Points] 2. Determine the singular points of the differential equation $2x(x-2)^2y'' + 3xy' + (x-2)y = 0$ and classify them as regular or irregular.

Solution $\quad P(x) = 2x(x-2)^2$. Let $P(x) = 0$, $\Rightarrow x = 0, 2$

are singular points. $\quad p(x) = \dfrac{Q(x)}{P(x)} = \dfrac{3x}{2x(x-2)^2} = \dfrac{3}{2(x-2)^2}$,

~~For $x_0 = 0$~~, and $\quad q(x) = \dfrac{R(x)}{P(x)} = \dfrac{x-2}{2x(x-2)^2} = \dfrac{1}{2x(x-2)}$.

For $x_0 = 0$, $\quad \lim\limits_{x \to 0} x \cdot p(x) = \lim\limits_{x \to 0} \dfrac{3x}{2(x-2)^2} = 0$,

$$\lim_{x \to 0} x^2 \cdot q(x) = \lim_{x \to 0} \frac{x^2}{2x(x-2)} = 0$$

Then $x = 0$ is a <u>regular</u> singular point.

For $x_0 = 2$, $\quad \lim\limits_{x \to 2} (x-2) \cdot p(x) = \lim\limits_{x \to 2} \dfrac{3(x-2)}{2(x-2)^2}$

$$= \lim_{x \to 2} \frac{3}{2(x-2)} = \infty$$

Then $x = 2$ is an <u>irregular</u> singular point.

[6 Points] 3. Find the general solution of the Cauchy-Euler equation $x^2y'' + xy' + y = 0$ for $x > 0$.

<u>Solution</u> Let $F(r) = r(r-1) + 1\cdot r + 1 = r^2 + 1 = 0$, we have $r = \pm i$ (so $\mu = 0$, $\nu = 1$).

Thus, $y = c_1 x^{\mu} \cos(\nu \ln x) + c_2 \cdot x^{\mu} \sin(\nu \ln x)$

$= c_1 \cos(\ln x) + c_2 \cdot \sin(\ln x)$, $x > 0$.

[6 Points] 4. Find the eigenvalues and normalized eigenfunctions of the eigenvalue problem $y'' + \lambda y = 0,\ y(0) = 0, y(1) = 0$.

<u>Solution</u> In the case where $\lambda > 0$, let $\lambda = \mu^2$, $(\mu > 0)$.

Since $r^2 + \mu^2 = 0$ has root $r = \pm i\mu$, we have

$y = c_1 \cos \mu x + c_2 \sin \mu x$ $y(0) = c_1 = 0$.

$y(1) = c_2 \sin \mu = 0$. $c_2 \neq 0 \Longrightarrow \mu = n\pi$, $\forall n \geq 1$

Thus, $\lambda_n = (n\pi)^2 = n^2 \cdot \pi^2$, $n = 1, 2, \cdots$ are eigenvalue with $\varphi_n = \sin(n\pi x)$ being the eigenfunction.

In the case where $\lambda = 0$, $y = c_1 x + c_2$. $y(0) = c_2 = 0$, and $y(1) = c_1 = 0$. Then $\lambda = 0$ is not an eigenvalue.

In the case where $\lambda < 0$, let $\lambda = -\mu^2$, $(\mu > 0)$.

Since $r^2 - \mu^2 = 0 \Longrightarrow r = \pm \mu$ Then

$y = c_1 \cosh(\mu x) + c_2 \sinh(\mu x)$. $y(0) = c_1 = 0$, and

$y(1) = c_2 \cdot \sinh(\mu) = 0 \Longrightarrow c_2 = 0$. Thus, $\lambda < 0$ is not an eigenvalue. Compared with $-[p(x)y']' + q(x)y = \lambda r(x) \cdot y$, we have $p(x) \equiv 1$, $q(x) \equiv 0$, $r(x) \equiv 1$. Then $\|\varphi_n\|_r^2 = \langle \varphi_n, \varphi_n \rangle$

$= \int_0^1 \sin^2(n\pi x)\,dx = \int_0^1 \frac{1 - \cos(2n\pi x)}{2} = \frac{1}{2}$. Thus, the normalized eigenfunctions $\psi_n(x) = \varphi_n / \|\varphi_n\| = \sqrt{2} \cdot \sin(n\pi x)$, $\forall n \geq 1$.

[6 Points] 5. Solve the nonhomogeneous boundary value problem $y''+2y=-x,\ y(0)=0, y(1)=0$ by means of an eigenfunction expansion.

Solution Compared with the general form

$$-[p(x)y']' + q(x)y = \mu \cdot r(x) \cdot y + f(x),$$

we see that $p(x)\equiv 1,\ q(x)\equiv 0,\ r(x)\equiv 1,\ \mu=2,\ f(x)=x$.

Thus, $$y = \sum_{n=1}^{\infty} b_n \phi_n(x), \quad \phi_n(x) = \sqrt{2}\sin(n\pi x).$$

Let $$f(x) = \sum_{n=1}^{\infty} c_n \cdot \phi_n(x)$$

Then $$c_n = \int_0^1 x\cdot\sqrt{2}\sin(n\pi x)\,dx = \sqrt{2}\cdot\int_0^1 x\sin(n\pi x)\,dx$$

$$= \sqrt{2}\cdot\frac{(-1)^{n+1}}{n\pi}, \quad \forall n \geq 1.$$

Thus, $$b_n = \frac{c_n}{\lambda_n - 2},$$ and

$$y = \sum_{n=1}^{\infty} \frac{1}{(n\pi)^2-2}\cdot\sqrt{2}\cdot\frac{(-1)^{n+1}}{n\pi}\cdot\sqrt{2}\sin(n\pi x)$$

$$= 2\cdot\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin(n\pi x)}{(n^2\pi^2-2)\cdot n\pi}.$$