Q2 (25).

$y' = 2x(1+y), \quad y(0) = 0.$

Solve.

(i) $y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\,dt$

$\Rightarrow y_n(x) = \int_0^x 2t(1 + y_{n-1}(t))\,dt, \quad \forall n \geq 1$

Therefore, $y_1(x) = \int_0^x 2t(1+0)\,dt = x^2$

$y_2(x) = \int_0^x 2t(1+t^2)\,dt = x^2 + \frac{1}{2}x^4$

$y_3(x) = \int_0^x 2t\,(1 + t^2 + \frac{1}{2}t^4)\,dt = x^2 + \frac{1}{2}x^4 + \frac{1}{6}x^6$

$y_4(x) = \int_0^x 2t\,(1 + t^2 + \frac{1}{2}t^4 + \frac{1}{6}t^6)\,dt = x^2 + \frac{1}{2}x^4 + \frac{1}{6}x^6 + \frac{1}{24}x^8.$

By induction, we have

$$y_n(x) = x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots + \frac{x^{2n}}{n!} = \sum_{k=1}^{n} \frac{x^{2k}}{k!}.$$

(ii) $\frac{dy}{1+y} = 2x\,dx$

$\Rightarrow \ln|1+y| = x^2 + C$

i.e. $y(x) = \tilde{C} e^{x^2} - 1$

Since $y(0) = 0 \Rightarrow \tilde{C} = 1$

$\Rightarrow y(x) = e^{x^2} - 1.$

Since $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$, it follows that

$e^{x^2} = \sum_{k=0}^{\infty} \frac{x^{2k}}{k!}.$ Therefore, $\lim_{n\to\infty} y_n(x) = e^{x^2} - 1.$ $\square$

section 70. Q2. (25).

(a) We prove this by contradiction. Assume that f is Lipschitz on the rectangle $|x| \le 1$, $0 \le y \le 1$. Then there exists a constant $L > 0$ such that for all $y_1, y_2 \in [0,1]$,

$$|\sqrt{y_1} - \sqrt{y_2}| \le L \cdot |y_1 - y_2|.$$

Now choose $y_1 = 0$ and $y_2 = t$, where $0 < t < 1$. Then both y_1 and y_2 lie in $[0,1]$, so the Lipschitz inequility gives

$$|\sqrt{t} - \sqrt{0}| \le L \cdot |t - 0|.$$

Hence, $\sqrt{t} \le L \cdot t$. Since $t > 0$, we divide both sides by t and obtain : $\frac{1}{\sqrt{t}} \le L$. However, $\frac{1}{\sqrt{t}} \to \infty$ as $t \to 0^+$

This is a contradiction. Thus, $f(x,y) = \sqrt{y}$ does not satisfy a Lipschitz condiction on the rectangle $|x| \le 1$, $0 \le y \le 1$.

(b). method 1:

Let $y_1, y_2 \in [c,d]$. Consider the function $g(y) = \sqrt{y}$. Since g is differentiable on (c,d) and continuous on $[c,d]$, the mean value theorem implies that there exists some $\xi \in (y_1, y_2)$ such that

$$g(y_1) - g(y_2) = g'(\xi) \cdot (y_1 - y_2)$$

$$\Rightarrow \quad |\sqrt{y_1} - \sqrt{y_2}| = |g'(\xi)| \cdot |y_1 - y_2|.$$

since $g'(y) = \frac{1}{2\sqrt{y}} \Rightarrow |g'(\xi)| = \frac{1}{2\sqrt{\xi}} \le \frac{1}{2\sqrt{c}}$.

Therefore, $|\sqrt{y_1} - \sqrt{y_2}| \le \frac{1}{2\sqrt{c}} \cdot |y_1 - y_2|$.

This show that $f(x,y) = \sqrt{y}$ satisfied a Lipschitz condition with $L = \frac{1}{2\sqrt{c}}$.

Method 2:

For any $y_1, y_2 \in [c,d] \Rightarrow |\sqrt{y_1} - \sqrt{y_2}| = \dfrac{|y_1 - y_2|}{\sqrt{y_1} + \sqrt{y_2}}$

since $y_1 \ge c$ and $y_2 \ge c$, we have

$$\sqrt{y_1} + \sqrt{y_2} \ge 2\sqrt{c}$$

Hence

$$|\sqrt{y_1} - \sqrt{y_2}| \le \frac{1}{2\sqrt{c}} \cdot |y_1 - y_2|$$

so f is Lipschitz on the rectange, with Lipschitz constant $L = \frac{1}{2\sqrt{c}}$.

$\boxed{}$.

6. (a) (25)

Sovle. $f(x,y) = y|y| = \begin{cases} y^2, & y \ge 0, \\ -y^2, & y < 0. \end{cases}$

Clearly, f is continuous for $\forall (x,y) \in \mathbb{R}^2$.

Moreover, $\partial_y f(x,y) = \begin{cases} 2y, & y > 0, \\ -2y, & y < 0. \end{cases}$

Note that $\partial_y^- f(x,0) = \lim_{y \to 0^-} \partial_y f(x,y)$

$= \lim_{y \to 0^-} (-2y)$

$= 0$

$\partial_y^+ f(x,0) = \lim_{y \to 0^+} \partial_y f(x,y)$

$= \lim_{y \to 0^+} (2y)$

$= 0$

and $\partial_y f(x,0) = 0$ for $\forall x \in \mathbb{R}$.

Therefore $\partial_y f$ is also continuous for all $(x,y) \in \mathbb{R}^2$.

By Thm A, it follows the desired result for $\forall (x,y) \in \mathbb{R}^2$.

(b) (25).

Solve. (1) $y_0 > 0$, $y' = y^2$

$$\Rightarrow \quad y = -\frac{1}{x+c}$$

By $y(x_0) = y_0$, it follows that $c = -\frac{1+y_0x_0}{y_0}$.

$$\Rightarrow \quad y = -\frac{y_0}{y_0(x-x_0)-1}.$$

(2) $y_0 = 0 \Rightarrow y' = 0 \Rightarrow y = c$

$y(x_0) = y_0 = 0 \Rightarrow c = 0$ Thus $y \equiv 0$.

(3) $y_0 < 0$, $y' = -y^2 \Rightarrow y = \frac{1}{x+c}$

Since $y(x_0) = y_0$, we have

$$c = \frac{1-x_0y_0}{y_0}$$

$$\Rightarrow \quad y(x) = \frac{y_0}{y_0(x-x_0)+1}.$$

The above discussions imply a unique solution is guarenteed for $y_0 = 0, >0, <0$. This completes the proof. ∎