68. 3(b). (30).

proof: Let x, y, z be the three sides of a trangle with fixed perimeter P, then we have:

$$x+y+z=P$$

Since $A=\sqrt{s\cdot(s-x)\cdot(s-y)\cdot(s-z)}$, where $s=\frac{x+y+z}{2}=\frac{P}{2}$

$\Rightarrow$ maximizing A is equivalent to maximizing

$$f(x,y,z)=(s-x)\cdot(s-y)\cdot(s-z)$$

subject to the constraint: $g(x,y,z)=x+y+z-P=0$

$$\Rightarrow \nabla g=\langle 1,1,1\rangle.$$

solve $\nabla f=\lambda\nabla g$, we have

$$-(s-y)\cdot(s-z)=\lambda$$
$$-(s-x)\cdot(s-z)=\lambda$$
$$-(s-x)\cdot(s-y)=\lambda$$

Hence; $(s-y)\cdot(s-z)=(s-x)\cdot(s-z)$.

since $s-z>0$ for a nondegenerate trangle, we obtain

$$s-y=s-x \quad\Rightarrow\quad x=y$$

Similarly. since $(s-x)\cdot(s-z)=(s-x)\cdot(s-y)$

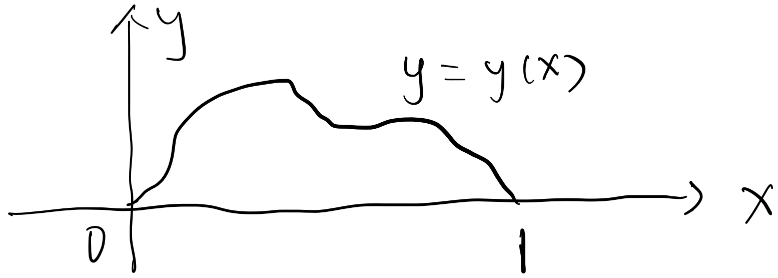
$$s-x>0 \Rightarrow s-z=s-y \Rightarrow z=y$$

Therefore, $x=y=z$.

Using the constrain $x+y+z=P$, we get

$$x=y=z=\frac{P}{3}.$$

4. (35).



Solve. Minimize $L := \int_0^1 \sqrt{1+(y')^2}\, dx$ with

condition $A = \int_0^1 y\, dx$ and $y(0) = 0 = y(1)$.

Letting $F = \sqrt{1+(y')^2} + \lambda y$, by the Euler formula,

we have

$$0 = \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y}$$

$$= \frac{d}{dx}\left(\frac{y'}{\sqrt{1+(y')^2}}\right) - \lambda.$$

Integrating both sides, we see that

$$\frac{y'}{\sqrt{1+(y')^2}} = \lambda x + C_1$$

$$\Rightarrow (y')^2 = \frac{(\lambda x + C_1)^2}{1-(\lambda x + C_1)^2}$$

$$\Rightarrow \frac{dy}{dx} = \pm \frac{\lambda x + C_1}{\sqrt{1-(\lambda x + C_1)^2}}$$

$$\Rightarrow y = \pm \int \frac{\lambda x + c_1}{\sqrt{1-(\lambda x+c_1)^2}}\, dx$$

$$\Rightarrow y = \mp \frac{1}{\lambda} \sqrt{1-(\lambda x+c_1)^2} + c_2$$

$$\Rightarrow (y-c_2)^2 \lambda^2 + (\lambda x + c_1)^2 = 1.$$

$$\Rightarrow (y-c_2)^2 + (x + c_3)^2 = \frac{1}{\lambda^2}.$$

This is a circle with radius $\frac{1}{\lambda}$.

Thus, the minimized curve is an arc of the above circle.

$\square$

7. (35) Show the geodesics on any cylinder of the form $g(x,z)=0$ make a constant angle with the y-axis.

Solve. The general formula of a cyclinder is

$$g(x,z) = x^2 + z^2 - r^2, \quad \forall y \in \mathbb{R}.$$

Letting $x = r\cos\theta$, $z = r\sin\theta$, we have

$g(x,z) = r^2 - r^2 = 0.$

Our starting integral comes from

$$ds = \sqrt{dx^2 + dy^2 + dz^2}$$

$$= \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 + \left(\frac{dz}{d\theta}\right)^2}\, d\theta$$

$$= \sqrt{r^2\sin^2\theta + r^2\cos^2\theta + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$= \sqrt{r^2 + (y')^2}\, d\theta.$$

To find the geodesic, it suffices to minimize the following arc length integral:

$$I = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + (y')^2}\, d\theta.$$

By Eurler's equation w.r.t. θ, we have

$$\frac{d}{d\theta}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} = 0.$$

Since $f = \sqrt{r^2 + (y')^2}$ only depends on y',

we can use case A:

$$\frac{d^2y}{d\theta^2} = 0 \quad \text{since } f_{y'y'} = 0 .$$

$\Rightarrow$ $y = c_1\theta + c_2$ are the geodesic for our problem. Since θ is the given angle of the cyclinder, the geodesics can be classified as helices.

By definition, helicies have a constant angle with the helical axis, the y-axis. Thus, the geodesics on any cylinder of the form $g(x,z)=0$ make a constant angle with y-axis. $\square$