1. a). $\dfrac{\sqrt{1+(y')^2}}{y}$

Solv. $I = \int_{x_1}^{x_2} \dfrac{\sqrt{1+(y')^2}}{y} \, dx$.

This belongs to case C. $(f(x,y,y') = f(y,y'))$.

By case C, $\dfrac{\partial f}{\partial y'} y' - f = c_1$.

Since $\dfrac{\partial f}{\partial y'} = \dfrac{y'}{y\sqrt{1+(y')^2}}$, it follows that

$$c_1 = \frac{(y')^2}{y\sqrt{1+(y')^2}} - \frac{\sqrt{1+(y')^2}}{y}$$

$$= -\frac{1}{y\sqrt{1+(y')^2}}$$

$$\Rightarrow (y')^2 = \frac{1-y^2c_1^2}{y^2c_1^2}$$

$$\Rightarrow c_1 y' = \pm \frac{\sqrt{1-y^2c_1^2}}{y}$$

$$\Rightarrow \pm c_1 \frac{y}{\sqrt{1-y^2c_1^2}} dy = dx$$

Integrating both sides, we have

$$x = \pm c_1 \int \frac{y}{\sqrt{1-y^2c_1^2}} dy$$

$$= \pm \frac{1}{2c_1} \int (1-y^2c_1^2)^{-\frac{1}{2}} d(c_1^2y^2)$$

$$= \mp \frac{1}{c_1}\sqrt{1-y^2c_1^2} + c_2$$

That is, $(x-c_2)^2 + y^2 = \frac{1}{c_1^2}$.

$\boxed{\text{Q}}$

(b) $y^2 - (y')^2$.

Solve. By case C, $\frac{\partial f}{\partial y'} y' - f = c_1$.

Since $\frac{\partial f}{\partial y'} = -2y'$, we have

$$(y')^2 + y^2 = -c_1$$

$$\overset{c_2 := -c_1}{\Longrightarrow} \quad \frac{dy}{dx} = \pm\sqrt{c_2 - y^2}$$

$$\Rightarrow \quad \frac{1}{\sqrt{c_2-y^2}}\, dy = \pm\, dx$$

$$\Rightarrow \quad x = \pm\int \frac{1}{\sqrt{c_2-y^2}}\, dy$$

$$= \pm \arcsin\left(\frac{y}{\sqrt{c_2}}\right) - c_3$$

$$\Rightarrow \quad y = \pm\sqrt{c_2}\sin(x+c_3).$$

$\boxed{\text{Q}}$.

2. $\int_0^4 [xy' - (y')^2]\, dx \qquad y(0)=0,\ y(4)=3.$

Solve. This is case A. $\frac{\partial f}{\partial y'} = c_1$.

Since $\frac{\partial f}{\partial y'} = x - 2y'$, we see that

$$\frac{dy}{dx} = \frac{1}{2}(x - c_1)$$

$$\Rightarrow \quad y = \frac{1}{4}x^2 - \frac{1}{2}c_1 x + c_2$$

By B. C. $\begin{cases} c_2 = 0 \\ \frac{1}{4}\cdot 4^2 - c_1 \cdot \frac{4}{2} = 3 \end{cases}$

$$\Rightarrow \quad c_1 = \frac{1}{2}$$

$$\Rightarrow \quad y = \frac{1}{4}x(x-1).$$

$\boxtimes$.

3.

proof: Now $f(x, y, y') = a(x)\cdot(y')^2 + 2b(x)\cdot yy' + c(x)\cdot y^2$

then we have: $f_y = 2b(x)\cdot y' + 2c(x)\cdot y$

$f_{y'} = 2a(x)\cdot y' + 2b(x)\cdot y$

substituting into $f_y - \frac{d}{dx}(f_{y'}) = 0$

$$\Rightarrow \quad 2by' + 2cy - \frac{d}{dx}(2ay' + 2by) = 0$$

$$\Rightarrow \quad by' + cy - (ay')' - (by)' = 0$$

since: $(by)' = b'y + by'$

$(ay')' = a'\cdot y' + ay''$

$$\Rightarrow \quad a(x)\cdot y'' + a'(x)\cdot y' + [b'(x) - c(x)]\cdot y = 0$$

This is a second order linear differential equation.

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