§10.5

6 (20). $f(x) = x$. $x \in [0,1]$.

In each of Problems 5 through 8, find the coefficients in the eigenfunction expansion $\sum_{n=1}^{\infty} a_n \phi_n(x)$ of the given function, using the normalized eigenfunctions of the BVP $y'' + \lambda y = 0, y'(0) = 0, y'(1) + y(1) = 0$:

Solve. Consider $\lambda > 0$, Letting $y = e^{rx}$, we have

$$r^2 = -\lambda \Rightarrow r = \pm\sqrt{\lambda}\, i$$

$$\Rightarrow y = c_1 \sin(\sqrt{\lambda} x) + c_2 \cos(\sqrt{\lambda} x).$$

$$y' = c_1 \sqrt{\lambda} \cos(\sqrt{\lambda} x) - c_2 \sqrt{\lambda} \sin(\sqrt{\lambda} x)$$

By BCs $\Rightarrow$
$$\begin{cases} c_1 = 0 \\ y'(1) + y(1) = c_2 (\cos\sqrt{\lambda} - \sqrt{\lambda} \sin\sqrt{\lambda}) = 0 \end{cases}$$

$\xrightarrow{c_2 \neq 0}$ (otherwise trivial soln.) $\cos\sqrt{\lambda_n} = \sqrt{\lambda_n} \sin\sqrt{\lambda_n}$ (*) (there are infinite many eigenvalues).

Thus for λ_n, the corresponding eigenfunction is $\phi_n = c_2 \cos\sqrt{\lambda_n} x$.

$$1 = \|\phi_n\|^2 = \langle \phi_n, \phi_n \rangle$$

$$= C_2^2 \int_0^1 \cos^2(\sqrt{\lambda_n} x)\, dx$$

$\cos^2 x = \frac{1+\cos(2x)}{2}$

$$= \frac{C_2^2}{2} \int_0^1 [1 + \cos(2\sqrt{\lambda_n} x)]\, dx$$

$$= \frac{C_2^2}{2} \left(1 + \frac{\sin(2\sqrt{\lambda_n})}{2\sqrt{\lambda_n}} \right)$$

$$= \frac{C_2^2}{2} \left(1 + \frac{\sin\sqrt{\lambda_n} \cos\sqrt{\lambda_n}}{\sqrt{\lambda_n}} \right)$$

$$= \frac{C_2^2}{2} (1 + \sin^2\sqrt{\lambda_n})$$

$$\Rightarrow \quad C_2 = \sqrt{\frac{2}{1+\sin^2\sqrt{\lambda_n}}}.$$

$$\Rightarrow \text{Normalized } \phi_n = \sqrt{\frac{2}{1+\sin^2\sqrt{\lambda_n}}} \cos(\sqrt{\lambda_n} x).$$

$$f(x) = \sum_{n=1}^{\infty} a_n \phi_n(x)$$

where $a_n = \langle f, \phi_n \rangle$

$$= C_2 \int_0^1 x \cos(\sqrt{\lambda_n} x)\, dx$$

$$= C_2 \int_0^1 \frac{1}{\sqrt{\lambda_n}} x \, d \sin(\sqrt{\lambda_n} x)$$

$$= \frac{C_2}{\sqrt{\lambda_n}}\left[x \sin(\sqrt{\lambda_n} x) \Big|_0^1 - \int_0^1 \sin(\sqrt{\lambda_n} x)\, dx \right]$$

$$= \frac{C_2}{\sqrt{\lambda_n}}\left(\sin\sqrt{\lambda_n} + \frac{1}{\sqrt{\lambda_n}} \cos(\sqrt{\lambda_n} x)\Big|_0^1 \right)$$

$$= \sqrt{\frac{2}{1+\sin^2\sqrt{\lambda_n}}} \cdot \frac{\sqrt{\lambda_n}\sin\sqrt{\lambda_n} + \cos\sqrt{\lambda_n} - 1}{\lambda_n}$$

By (*) $\Longrightarrow$ $$a_n = \frac{\sqrt{2}(2\sqrt{\lambda_n}\sin\sqrt{\lambda_n} - 1)}{\lambda_n(1+\sin^2\sqrt{\lambda_n})}.$$

$\boxtimes$

10. $y'' + 2y = -x, \quad y(0) = 0, \; y'(1) = 0.$

Solve. Recall that $L[y] = -(p(x)y')' + q(x)y$

$= \mu r(x) y + f(x).$

$\Rightarrow p(x) = 1, \; q(x) = 0, \; r(x) = 1, \; \mu = 2, \; f(x) = x,$

Consider $L[y] = \lambda r(x) y.$

$\Rightarrow y'' + \lambda y = 0$

By the previous question,

$y = c_1 \sin(\sqrt{\lambda} x) + c_2 \cos(\sqrt{\lambda} x)$

BCs.

$$\Longrightarrow \begin{cases} C_2 = 0 \\ y'(1) = c_1 \sqrt{\lambda} \cos\sqrt{\lambda} = 0 . \end{cases}$$

$$\overset{C_1 \neq 0}{\underset{\lambda > 0}{\Longrightarrow}} \quad \cos\sqrt{\lambda_n} = 0$$

$$\Longrightarrow \quad \lambda_n = \frac{(2n+1)^2 \pi^2}{4}, \quad n \in \mathbb{N}.$$

$$\Updownarrow$$

$$\phi_n = c_1 \sin\sqrt{\lambda_n} x .$$

$$1 = \|\phi_n\|^2 = \langle \phi_n, \phi_n \rangle$$

$$= c_1^2 \int_0^1 \sin^2(\sqrt{\lambda_n} x) \, dx$$

$$\overset{\sin^2 x = \frac{1-\cos(2x)}{2}}{=\!=\!=} \quad \frac{c_1^2}{2} \int_0^1 [1 - \cos(2\sqrt{\lambda_n} x)] dx$$

$$= \frac{c_1^2}{2} \cdot \frac{2\sqrt{\lambda_n} - \sin(2\sqrt{\lambda_n})}{2\sqrt{\lambda_n}}$$

$$\overset{\cos\sqrt{\lambda_n} = 0}{\underset{\sin(2\sqrt{\lambda_n}) = 2\sin\sqrt{\lambda_n}\cos\sqrt{\lambda_n}}{=\!=\!=}} \quad \frac{c_1^2}{2}$$

$\Rightarrow \quad c_1 = \sqrt{2}.$

Now let $y = \sum_{n=1}^{\infty} b_n \phi_n$ where

$$b_n = \frac{c_n}{\lambda_n - \mu}, \quad \lambda_n \neq \mu.$$

$$\Rightarrow \quad \frac{f(x)}{r(x)} = f(x) = x = \sum_{n=1}^{\infty} c_n \phi_n$$

$$\Rightarrow \quad c_n = \langle x, \phi_n \rangle$$

$$= \int_0^1 x c_1 \sin(\sqrt{\lambda_n} x) dx$$

$$= -\frac{c_1}{\sqrt{\lambda_n}} \int_0^1 x \, d\cos(\sqrt{\lambda_n} x)$$

$$= -\frac{c_1}{\sqrt{\lambda_n}} \left[x \cos(\sqrt{\lambda_n} x) \Big|_0^1 - \int_0^1 \cos(\sqrt{\lambda_n} x) dx \right]$$

$$= -\frac{c_1}{\sqrt{\lambda_n}} \cos\sqrt{\lambda_n} + \frac{1}{\lambda_n} \sin(\sqrt{\lambda_n} x) \Big|_0^1$$

$$\overset{\cos\sqrt{\lambda_n} = 0}{=} \frac{c_1}{\lambda_n} \sin\sqrt{\lambda_n}$$

$$= \frac{\sqrt{2}}{\lambda_n} \sin\sqrt{\lambda_n}$$

$\therefore \quad b_n = \dfrac{c_n}{\lambda_n - 2} = \dfrac{\sqrt{2}\sin\sqrt{\lambda_n}}{\lambda_n(\lambda_n - 2)}$

$\Rightarrow y(x) = 2\sum_{n=1}^{\infty} \dfrac{\sin\sqrt{\lambda_n}}{\lambda_n(\lambda_n-2)} \sin(\sqrt{\lambda_n}x)$.

☒

16. $y'' + \mu y = -f(x), \quad y'(0) = 0, \quad y'(1) = 0$.

In each of Problems 14 through 17, determine a formal eigenfunction series expansion for the solution of the given problem. Assume that f satisfies the conditions of Theorem 10.5.1. State the values of μ for which the solution exists.

Solve. $L[y] = -(p(x)y')' + q(x)y = \mu r(x)y + f(x)$

$\Rightarrow p(x) = 1, \; q(x) = 0, \; r(x) = 1$.

Consider $L[y] = \lambda r(x) y$

$\Rightarrow y'' + \lambda y = 0$.

As we know $\lambda_n \in \mathbb{R}_+ \setminus \{0\}$.

$y = c_1 \sin(\sqrt{\lambda}x) + c_2 \cos(\sqrt{\lambda}x)$

By BCs. $y' = c_1\sqrt{\lambda}\cos(\sqrt{\lambda}x) - c_2\sqrt{\lambda}\sin(\sqrt{\lambda}x)$

$\Rightarrow$

$$\begin{cases} c_1 = 0 \\ y'(1) = -c_2\sqrt{\lambda}\sin\sqrt{\lambda} = 0 \end{cases}$$

$$\overset{c_2 \neq 0}{\underset{\sqrt{\lambda}\neq 0}{\Longrightarrow}} \sin\sqrt{\lambda} = 0$$

$$\Rightarrow \lambda_n = n^2\pi^2, \quad n \in \mathbb{N}.$$

$$\updownarrow$$

$$\varphi_n(x) = c_2\cos(\sqrt{\lambda_n}\,x).$$

$$1 = \|\varphi_n\| = \langle\varphi_n, \varphi_n\rangle$$

$$= c_2^2\int \cos^2(\sqrt{\lambda_n}\,x)\,dx$$

$$\overset{\cos^2 x = \frac{1+\cos(2x)}{2}}{=\!=\!=} \frac{c_2^2}{2}\int_0^1 [1+\cos(2\sqrt{\lambda_n}\,x)]\,dx$$

$$= \frac{c_2^2}{2}\left(1 + \frac{\sin\sqrt{\lambda_n}\cos\sqrt{\lambda_n}}{\sqrt{\lambda_n}}\right)$$

$$\overset{\sin\sqrt{\lambda_n}=0}{=\!=} \frac{c_2^2}{2}$$

$$\Rightarrow c_2 = \sqrt{2}, \quad \varphi_n(x) = \sqrt{2}\cos(\sqrt{\lambda_n}\,x).$$

Let $y = \sum_{n=1}^{\infty} b_n\varphi_n$ where $b_n = \dfrac{c_n}{\lambda_n - \mu}$

$$\Rightarrow \frac{f(x)}{r(x)} = f(x) = \sum_{n=1}^{\infty} c_n \phi_n$$

$$\Rightarrow c_n = \langle f, \phi_n \rangle$$

$$= \int_0^1 f(x)\, \phi_n(x)\, dx$$

$$= \sqrt{2} \int_0^1 f(x) \cos(\sqrt{\lambda_n}\, x)\, dx.$$

Since f satisfies Thm 10.5.1, c_n is unique when $\lambda_n \neq \mu$. This implies the solution exists for $\mu \neq n^2\pi^2$, $n \in \mathbb{N}$.

$$\Rightarrow b_n = \frac{\sqrt{2}}{\lambda_n - \mu} \int_0^1 f(x) \cos(\sqrt{\lambda_n}\, x)\, dx$$

$$\Rightarrow y(x) = 2 \sum_{n=1}^{\infty} \frac{1}{\lambda_n - \mu} \left[\int_0^1 f(x) \cos(\sqrt{\lambda_n}\, x)\, dx \right] \cos(\sqrt{\lambda_n}\, x).$$

$\mu \neq n^2\pi^2$, $n \in \mathbb{N}$. ◻

20. $y'' + \pi^2 y = a, \quad y'(0) = 0, \; y'(1) = 0.$

In each of Problems 18 through 21, determine whether there is any value of the constant a for which the problem has a solution. Find the solution for each such value.

Solve. Consider $y'' + \pi^2 y = 0$.

Let $y = e^{rx}$ $\Rightarrow$

$r^2 + \pi^2 = 0 \Rightarrow r = \pm\sqrt{\pi}\, i$.

$\Rightarrow y = C_1 \cos(\sqrt{\pi}x) + C_2 \sin(\sqrt{\pi}x)$

$y' = \sqrt{\pi}\left[-C_1 \sin(\sqrt{\pi}x) + C_2 \cos(\sqrt{\pi}x)\right]$

Bcs. $\Rightarrow$ $\begin{cases} C_2 = 0 \\ -C_1 \pi \sin \pi = 0 \end{cases}$ (No restriction on C_1)

$\Rightarrow y(x) = C_1 \cos(\pi x)$ is the solution to the homogeneous case.

Let $y_p = k$ be the particular solution to nonhomogeneous equation.

$y_p'' + \pi^2 y_p = a \Rightarrow \pi^2 k = a \Rightarrow k = \frac{a}{\pi^2}$.

$\Rightarrow y(x) = C_1 \cos(\pi x) + \frac{a}{\pi^2}$, $\forall C_1 \in \mathbb{R}$.

$\boxtimes$

2. Consider the boundary value problem

$$-(xy')' = \lambda xy,$$
$$y,\ y' \text{ bounded as } x \to 0, \qquad y'(1) = 0.$$

(a) Show that $\lambda_0 = 0$ is an eigenvalue of this problem corresponding to the eigenfunction $\phi_0(x) = 1$. If $\lambda > 0$, show formally that the eigenfunctions are given by $\phi_n(x) = J_0(\sqrt{\lambda_n}\,x)$, where $\sqrt{\lambda_n}$ is the nth positive root

(in increasing order) of the equation $J_0'(\sqrt{\lambda}) = 0$. It is possible to show that there is an infinite sequence of such roots.
(b) Show that if $m, n = 0, 1, 2, \ldots$, then

$$\int_0^1 x\phi_m(x)\phi_n(x)\,dx = 0, \qquad m \neq n.$$

Solve. (a) For $\lambda_0 = 0$, $-(xy')' = 0$

$\Rightarrow \quad y' + xy'' = 0.$

Let $y = e^{rx} \Rightarrow x^{r-1}(r^2 + r - r) = 0, \ \forall x \in \mathbb{R},$

$\Rightarrow r = 0$ is a double root.

$\Rightarrow \quad y(x) = C_1 x^0 + C_2 x^0 \ln(x)$

$= C_1 + C_2 \ln x. \quad \Rightarrow y'(x) = \frac{C_2}{x}$

By BC. $\underset{y'(1)=0}{\Longrightarrow}$ $C_2 = 0$

$\Rightarrow \phi(x) = C_1$. Clearly normalized one is

$\phi(x) = 1$,

Now, consider $\lambda > 0$, $-(xy')' = \lambda x y$. $(*)$

$p(x) = x, q(x) = 0, r(x) = x$. Note that $p(0) = r(0) = 0$,

Let $t = \sqrt{\lambda} x$, by the chain rule,

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \sqrt{\lambda} \frac{dy}{dt},$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right) \cdot \frac{dt}{dx} = \lambda \frac{d^2y}{dt^2}.$$

By $(*)$, $xy'' + y' + \lambda x y = 0$

change to t $\Rightarrow$

$$t \frac{d^2y}{dt^2} + \frac{dy}{dt} + t y = 0.$$

$$\Rightarrow t^2 \frac{d^2y}{dt^2} + t \frac{dy}{dt} + t^2 y = 0,$$ which is

Bessel's equation of order 0 with the

Solution in the form of

$y(t) = c_1 J_0(t) + c_2 Y_0(t), \quad t>0.$

i.e., $y(x) = c_1 J_0(\sqrt{\lambda} x) + c_2 Y_0(\sqrt{\lambda} x)$, where

$$J_0(\sqrt{\lambda} x) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m (\sqrt{\lambda} x)^{2m}}{2^{2m} \cdot (m!)^2}, \quad x>0$$

and $$Y_0(\sqrt{\lambda} x) = \frac{2}{\pi}\left[(\gamma + \ln \frac{\sqrt{\lambda} x}{2}) J_0(\sqrt{\lambda} x) + \sum_{m=1}^{\infty} \frac{(-1)^{m+1} H_m (\sqrt{\lambda} x)^{2m}}{2^{2m} (m!)^2} \right]$$

With $H_m = 1 + \frac{1}{2} + \cdots + \frac{1}{m}$ and $\gamma = \lim_{m\to\infty} (H_m - \ln m)$.

Since $\lim_{x\to 0} Y(x)$ is unbounded while y, y' are bounded as $x \to 0$, we choose $c_2 = 0$. That is,

$y(x) = c_1 J_0(\sqrt{\lambda} x)$

By $\overset{y'(1)=0}{\Longrightarrow}$ $c_1 J_0'(\sqrt{\lambda}) = 0$

$\overset{c_1 \neq 0}{\Longrightarrow}$ $J_0'(\sqrt{\lambda}) = 0.$

Thus, $\phi_n(x) = J_0(\sqrt{\lambda_n} x)$, where $\sqrt{\lambda_n}$ is the nth positive root of $J_0'(\sqrt{\lambda_n}) = 0$, $\forall n \geq 1$.

(b). Goal: $\int_0^1 x\, \phi_m(x) \phi_n(x)\, dx$

$$= \int_0^1 x J_0(\sqrt{\lambda_m} x) J_0(\sqrt{\lambda_n} x)\, dx = 0.$$

Note that $[J_0(\sqrt{\lambda_m} x)]' = -\sqrt{\lambda_m} J_1(\sqrt{\lambda_m} x)$ (*)

and $[x J_1(\sqrt{\lambda_m} x)]' = \sqrt{\lambda_m}\, x J_0(\sqrt{\lambda_m} x)$.

Step 1. Replace $x J_0(\sqrt{\lambda_m} x)$ by $\frac{1}{\sqrt{\lambda_m}}[x J_1(\sqrt{\lambda_m} x)]'$.

Then $\int_0^1 \frac{1}{\sqrt{\lambda_m}} [x J_1(\sqrt{\lambda_m} x)]' J_0(\sqrt{\lambda_n} x)\, dx$

$$= \frac{1}{\sqrt{\lambda_m}} \int_0^1 J_0(\sqrt{\lambda_n} x)\, d[x J_1(\sqrt{\lambda_m} x)]$$

By (*)

$$= \frac{1}{\sqrt{\lambda_m}} J_0(\sqrt{\lambda_n} x)\, x J_1(\sqrt{\lambda_m} x) \Big|_0^1$$

$$+ \frac{\sqrt{\lambda_n}}{\sqrt{\lambda_m}} \int_0^1 x J_1(\sqrt{\lambda_n} x) J_1(\sqrt{\lambda_m} x)\, dx$$

$$= -\frac{1}{\lambda_m} x J_0'(\sqrt{\lambda_m} x) J_0(\sqrt{\lambda_n} x) \Big|_0^1$$

$$= -\frac{1}{\sqrt{\lambda_m}} J_0'(\sqrt{\lambda_m}) J_0(\sqrt{\lambda_n}) = 0 \text{ since } J_0'(\sqrt{\lambda_m}) = J_0'(\sqrt{\lambda_n}) = 0$$

by part (a).

Therefore, $\int_0^1 x\,\phi_m(x)\,\phi_n(x)\,dx$

$$= \frac{\sqrt{\lambda_n}}{\sqrt{\lambda_m}} \int_0^1 x\,J_1(\sqrt{\lambda_n}x)\,J_1(\sqrt{\lambda_m}x)\,dx. \quad (**)$$

Step 2. Consider $\int_0^1 x\,J_0(\sqrt{\lambda_m}x)\,J_0(\sqrt{\lambda_n}x)\,dx$.

Replace $x J_0(\sqrt{\lambda_n}x)$ by $\frac{1}{\sqrt{\lambda_n}}[x J_1(\sqrt{\lambda_n}x)]'$,

by the similar process, we have

$$\int_0^1 x\,\phi_m(x)\,\phi_n(x)\,dx = \frac{\sqrt{\lambda_m}}{\sqrt{\lambda_n}} \int_0^1 x\,J_1(\sqrt{\lambda_n}x)\,J_1(\sqrt{\lambda_m}x)\,dx.$$

This, combined with $(**)$, yields that

$$\left(\frac{\sqrt{\lambda_m}}{\sqrt{\lambda_n}} - \frac{\sqrt{\lambda_n}}{\sqrt{\lambda_m}}\right) \int_0^1 x\,J_1(\sqrt{\lambda_n}x)\,J_1(\sqrt{\lambda_m}x)\,dx = 0$$

Since $\lambda_n \neq \lambda_m$, $n \neq m$, it follows that

$\int_0^1 x\,J_1(\sqrt{\lambda_n}x)\,J_1(\sqrt{\lambda_m}x)\,dx = 0$, and hence,

$$\int_0^1 x\,\phi_m(x)\,\phi_n(x)\,dx = \frac{\sqrt{\lambda_m}}{\sqrt{\lambda_n}} \int_0^1 x\,J_1(\sqrt{\lambda_n}x)\,J_1(\sqrt{\lambda_m}x)\,dx$$

$$= 0, \quad n \neq m.$$

$\boxtimes$