AS - 3. Total 100.

§10.4 Q20 (25); Else (15).

§10.3. Q10. $y'' + 3y = \cos x$, $y'(0) = y'(\pi) = 0$.

solve: we first consider the homogeneous equation:

$y'' + 3y = 0$. characteristic equation: $r^2 + 3 = 0$

so $r = \pm\sqrt{3}\, i$. Thus the general solution is:

$$y_c(x) = A \cdot \sin\sqrt{3}x + B\cos\sqrt{3}x.$$

Assume $y_p(x) = a \cdot \cos x + b \sin x$ is a particular sol.

$\Rightarrow y_p'(x) = -a\sin x + b\cos x$

$y_p''(x) = -a\cos x - b\sin x$

substituting $y_p(x)$ into the non-homogenous eqution.

$\Rightarrow -a\cos x - b\sin x + 3a\cos x + 3b \cdot \sin x = \cos x$

$\Rightarrow 2a = 1$ and $2b = 0$ $\Rightarrow a = \frac{1}{2}$ and $b = 0$

Thus, the solution of non-homogeneous equation is:

$$y(x) = A\sin\sqrt{3}x + B\cos\sqrt{3}x + \frac{1}{2}\cos x.$$

Applying $y'(0) = 0$ and $y'(\pi) = 0$

$\Rightarrow \begin{cases} \sqrt{3}A = 0 \\ -\sqrt{3} \cdot B \cdot \sin(\sqrt{3}\pi) = 0 \end{cases} \Rightarrow \begin{cases} A = 0 \\ B = 0 \end{cases}$

So the solution to the boundary value problem is:

$$y(x) = \frac{1}{2} \cdot \cos x$$

√Π

12. $x^2y''+3xy'+y=x^2 \quad (*)$, $y(1)=0=y(e)$.

Solve. Consider homo. eq. $x^2y''+3xy'+y=0 \quad (**)$, which is one kind of Cauchy-Euler eq..

Thus, assume $y=x^r$ is the soln. of $(**)$.

Indicial eq. : $0=F(r)=r(r-1)+\alpha r+\beta$

$$= r^2-r+3r+1$$

$$=(r+1)^2$$

$$\Rightarrow r_1=r_2=-1.$$

$\Rightarrow y(x)=\frac{1}{|x|}(c_1+c_2\ln|x|)$ is the soln. to $(**)$, $\forall x\neq 0$.

Next, for $(*)$. Let $y_p := ax^2$.

$$\Rightarrow y_p'=2ax,$$

$$y_p''=2a.$$

Substituting them into $(*)$, we have

$$2ax^2+6ax^2+ax^2=x^2 \Rightarrow a=\frac{1}{9}.$$

$\Rightarrow y(x)=\frac{1}{|x|}(c_1+c_2\ln|x|)+\frac{1}{9}x^2$ is the soln. to $(*)$, $\forall x\neq 0$.

B.C.
$$\begin{cases} y(1)=0 \\ y(e)=0 \end{cases} \Rightarrow \begin{cases} c_1=-\frac{1}{9} \\ c_2=\frac{1}{9}(1-e^3) \end{cases}$$

$$\Rightarrow y(x)=\frac{-1}{|x|}\left[\frac{1}{9}+\frac{1}{9}(e^3-1)\ln|x|\right]+\frac{1}{9}x^2, \ \forall x\neq 0.$$

$\square$

22. $y'' + \lambda y = 0$, $y'(0) = 0 = y(L)$.

Solve. (i) $\lambda > 0$ $\lambda = \mu^2$ $(\mu > 0)$

$r^2 = -\mu^2 \Rightarrow r = \pm \mu i$.

$\Rightarrow y = C_1 \cos(\mu x) + C_2 \sin(\mu x)$

$y' = -\mu C_1 \sin(\mu x) + \mu C_2 \cos(\mu x)$

$\begin{cases} y'(0) = 0 \\ y(L) = 0 \end{cases} \Rightarrow \begin{cases} C_2 = 0, \\ C_1 \cos(\mu L) = 0. \end{cases}$

If $C_1 \neq 0$, then $\mu L = \frac{\pi}{2} + n\pi$, $n \in \mathbb{N}_+$ $(0, 1, 2, \cdots)$

$\Rightarrow \lambda_n = \frac{(2n+1)^2 \pi^2}{4L^2}$, $n \in \mathbb{N}_+$

$\updownarrow$

$\phi_n(x) = C_1 \cos\left(\frac{\pi(2n+1)}{2L} x\right)$, $\forall C_1 \in \mathbb{R} \setminus \{0\}$.

(ii) $\lambda = 0$, $y(x) = C_1 x + C_2$

$\begin{cases} y'(0) = 0 \\ y(L) = 0 \end{cases} \Rightarrow \begin{cases} C_1 = 0 \\ C_2 = 0 \end{cases}$

$\therefore \nexists$ nontrivial solns $\Rightarrow$ 0 is not an eigenvalue.

(iii) $\lambda < 0$. $\lambda = -\mu^2$ $(\mu > 0)$

$r^2 = \mu^2 \Rightarrow r = \pm \mu$.

$\Rightarrow y = C_1 \cosh(\mu x) + C_2 \sinh(\mu x)$.

$y' = \mu C_1 \sinh(\mu x) + \mu C_2 \cosh(\mu x)$.

$\begin{cases} y'(0) = 0 \\ y(L) = 0 \end{cases} \Rightarrow \begin{cases} C_2 = 0 \\ C_1 \cosh(\mu L) = 0 \end{cases}$

If $c_1 \neq 0$, then $\cosh(\mu L) = 0$, a Contradiction Since

$\cosh(x) > 0$, $\forall x \in \mathbb{R}$.

$\Rightarrow$ $c_1 = 0$. i.e., $\nexists$ nontrivial solns.

Thus, there are no negative eigenvalues. $\boxtimes$

24. $y'' - \lambda y = 0$, $y(0) = y'(L) = 0$.

Solve. (i) $\lambda > 0$ $\lambda = \mu^2$ $(\mu > 0)$

$r^2 = + \mu^2$ $\Rightarrow$ $r = \pm \mu$

$$\Rightarrow \quad y = c_1 \cosh(\mu x) + c_2 \sinh(\mu x)$$

$$y' = \mu c_1 \sinh(\mu x) + \mu c_2 \cosh(\mu x)$$

$$\begin{cases} y(0) = 0 \\ y'(L) = 0 \end{cases} \Rightarrow \begin{cases} c_1 = 0 \\ c_2 \cosh(\mu L) = 0 \end{cases}$$

If $c_2 \neq 0$, then $\cosh(\mu L) = 0$, a contradiction.

$\Rightarrow$ $c_2 = 0$. $\nexists$ nontrivial solns. $\Rightarrow$ no positive eigenvalues.

(ii) $\lambda = 0$, $y = c_1 x + c_2$

$$\begin{cases} y(0) = 0 \\ y'(L) = 0 \end{cases} \Rightarrow \begin{cases} c_2 = 0 \\ c_1 = 0 \end{cases}$$ 0 is not an eigenvalue.

(iii) $\lambda < 0$, $\lambda := -\mu^2$ $(\mu > 0)$

$\Rightarrow$ $r^2 = -\mu^2$ $\Rightarrow$ $r = \pm \mu i$.

$\Rightarrow$ $y = c_1 \cos(\mu x) + c_2 \sin(\mu x)$.

$$\begin{cases} y(0)=0 \\ y'(L)=0 \end{cases} \Rightarrow \begin{cases} c_1 = 0 \\ c_2 \cos(uL) = 0 \end{cases}$$

$$\overset{c_2 \neq 0}{\Rightarrow} \quad uL = \frac{1+2n}{2}\pi, \quad n \in \mathbb{N}_+.$$

$$\Rightarrow \lambda_n = -\left[\frac{(1+2n)\pi}{2L}\right]^2, \quad n \in \mathbb{N}_+.$$

Corresponding $\updownarrow$
eigenfunction: $\phi_n(x) = c_2 \sin\left(\frac{(1+2n)\pi}{2L}x\right), \ \forall c_2 \in \mathbb{R} \setminus \{0\}.$

$\square$

§10.4 2. $y'' + \lambda y = 0$, $y'(0) = 0 = y(1)$.

Solve. By the similar analysis process of §10.3 20.,

there are only positive eigenvalues, that is,

$\lambda > 0$, and $\lambda_n = \frac{(2n+1)^2 \pi^2}{4}$, $n \in \mathbb{N}_+$.

The corresponding eigenfunction is

$$\phi_n(x) = c_1 \cos\left(\frac{(2n+1)\pi}{2}x\right).$$

$$1 = \|\phi_n\|^2 = \langle \phi_n, \phi_n \rangle$$

$$= c_1^2 \int_0^1 \cos^2(\sqrt{\lambda_n}\, x)\, dx$$

$$\overset{\cos^2 x = \frac{1-\cos(2x)}{2}}{=} \frac{c_1^2}{2} \int_0^1 [1 - \cos(2\sqrt{\lambda_n}\, x)]\, dx$$

$$= \frac{1}{2}c_1^2 + \frac{c_1^2}{4\sqrt{\lambda_n}} \sin(\sqrt{2\lambda_n}\, x) \Big|_0^1$$

$$= \frac{1}{2} c_1^2$$

$\Rightarrow \quad C_1 = \pm\sqrt{2}$

$\therefore \quad \phi_n(x) = \pm\sqrt{2}\cos\left(\frac{(2n+1)\pi}{2}x\right)$ is the normalized eigenfunction $\square$

20. $y'' + \lambda y = 0, \quad \alpha y(0) + y'(0) = 0, \quad y(1) = 0.$

Solve. (a) Let $\lambda = \mu^2$, $\mu > 0$. Then

$$y(x) = C_1\cos(\mu x) + C_2\sin(\mu x).$$

$$\xrightarrow{B.C.} \begin{cases} \alpha C_1 + C_2\mu = 0, \\ \cos(\mu)C_1 + \sin(\mu)C_2 = 0. \end{cases}$$

To have nonzero soln. for (C_1, C_2), we need

$$\begin{vmatrix} \alpha & \mu \\ \cos\mu & \sin\mu \end{vmatrix} = 0$$

$$\Rightarrow \quad \alpha\sin\mu - \mu\cos\mu = 0 \xrightarrow{\cos\mu \neq 0} \mu = \alpha\tan\mu \quad (1).$$

Case (i) $\alpha \neq 0$, (1) has infinite many solutions $\mu_1, \mu_2, \cdots$, and hence, $\lambda_i = \mu_i^2$, $i = 1, 2, \cdots$ is an infinite sequence of positive eigenvalues.

Case (ii) $\alpha = 0$, $\Rightarrow C_2 = 0$, we have $\cos\mu = 0$,

$\Rightarrow \mu_i = \frac{(2i-1)}{2}\pi$, $i \geq 1$, and hence, $\lambda_i = \mu_i^2$ is an infinite sequence of positive eigenvalues.

(b) $\alpha < 1$. We first consider the case where $\lambda = 0$.

$$y'' = 0 \Rightarrow y = C_1 x + C_2$$

$$\Rightarrow \begin{cases} \alpha C_2 + C_1 = 0 \\ C_1 + C_2 = 0 \end{cases} \Rightarrow (\alpha - 1) C_2 = 0 \overset{\alpha<1}{\Longrightarrow} C_2 = 0 = C_1.$$

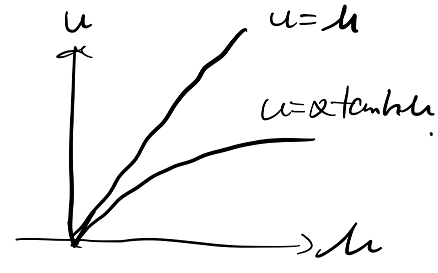
Thus $\lambda = 0$ is not the eigenvalue.

Secondly, we consider the case where $\lambda < 0$. Let $\lambda = -\mu^2$, $\mu > 0$. Then $y(x) = C_1 \sinh(\mu x) + C_2 \cosh(\mu x)$.

$$\overset{B.C.}{\Longrightarrow} \begin{cases} \alpha C_2 + C_1 \mu = 0 \\ \sinh(\mu) C_1 + \cosh(\mu) C_2 = 0 \end{cases}.$$

$$\overset{\binom{C_1}{C_2} \neq \binom{0}{0}}{\Longrightarrow} \begin{vmatrix} \alpha & c_1 \\ \sinh\mu & \cosh\mu \end{vmatrix} = \mu \cosh\mu - \alpha \sinh\mu = 0$$

$$\underset{\cosh\mu \neq 0}{\overset{\alpha \neq 0}{\Longrightarrow}} \mu = \alpha \tanh(\mu), \quad \mu > 0.$$



Since $\left.\dfrac{d}{d\mu}(\alpha \tanh(\mu))\right|_{\mu=0} = \alpha < 1$

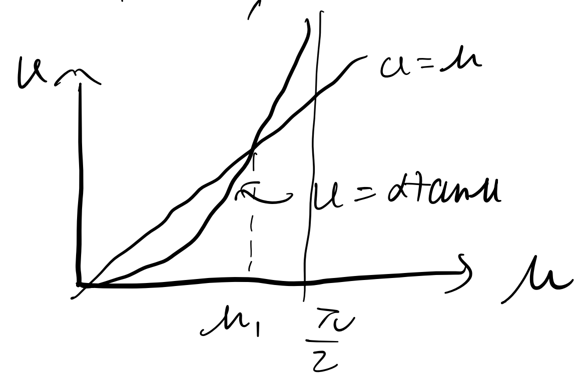
$\mu = \alpha \tanh(\mu)$ has no positive solution for μ, and hence, the problem has no negative eigenvalue.

Thus, if $\alpha < 1$, all eigenvalues are positive.

As discussed in (a) when $\alpha \neq 0$, the positive eigenvalues $\lambda_i = \mu_i^2$ and μ_i are roots of $\mu = \alpha \tan\mu$.

The first eigenvalue is dominated by $\lambda_1 = u_1^2$.

Note that $\frac{d}{du}(\alpha \tan u)\Big|_{u=0} = \alpha$, we see that as $\alpha \to 1^-$, $u_1 \to 0$, and hence, $\lambda_1 = u_1^2 \to 0$.



(c) $\lambda = 0$, as discussed in (b), $y = c_1 x + c_2$

$$\Rightarrow \begin{cases} \alpha c_2 + c_1 = 0 \\ c_1 + c_2 = 0 \end{cases} \Rightarrow (\alpha - 1) c_2 = 0$$

$\xrightarrow{c_2 \neq 0} \alpha - 1 = 0$. That is, $\alpha = 1$.

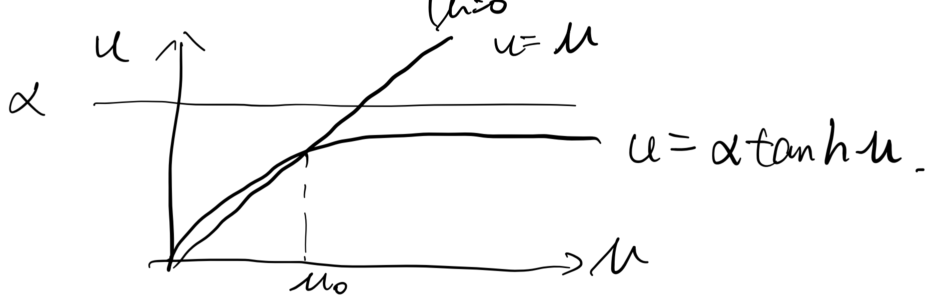
Otherwise $c_1 = 0$

Thus, $\lambda = 0$ is an eigenvalue only if $\alpha = 1$.

(d) If $\alpha > 1$. Let $\lambda = -u^2$, as discussed in (b),

$u = \alpha \tanh u$, $u > 0$. Note that

$\frac{d}{du}(\alpha \tanh(u))\Big|_{u=0} = \alpha > 1$ and $\tanh u \nearrow u$.



Thus, $\mu = \alpha \tanh \mu$ has only one positive solution μ_0, and hence, $\lambda = \mu_0^2$ is the only negative eigenvalue.

Since μ_0 increase as α increases, $\lambda = -\mu_0^2$ decreases as α increases.

$\square$