Total 100.

§ 9.5 Q2 (14), Q4 (10), Q8 (14).

§ 9.6 Q2 (8), Q6 (12), Q14 (14), Q16 (14),

§ 9.7 Q2 (14).

§ 9.2. 2. $x^2y'' + xy' + (x^2 - \frac{1}{9})y = 0.$ (*)

Solve. $P(x) = x^2$, $Q(x) = x$, $R(x) = x^2 - \frac{1}{9}$. Clearly, $P(0) = 0$.

Since $\lim_{x\to 0} x \cdot \frac{Q(x)}{P(x)} = 1 < \infty$, and

$\lim_{x\to 0} x^2 \frac{R(x)}{P(x)} = -\frac{1}{9} < \infty$,

$x\frac{P}{Q}$ and $x^2\frac{R}{P}$ are analytic at $x=0$. Thus 0 is a regular singular point.

Let $y = x^r \sum_{n=0}^{\infty} a_n x^n \Rightarrow y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$, $y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$.

It follows from (*) that

$$a_0[r(r-1)+r-\frac{1}{9}]x^r + a_1[r(r+1)+(r+1)-\frac{1}{9}]x^{r+1}$$
$$+ \sum_{n=2}^{\infty} x^{n+r}\{a_n[(n+r)(n+r-1)+(n+r)-\frac{1}{9}] + a_{n-2}\} = 0.$$

Assuming $a_0 \neq 0$, we see the indicial equation

$$F(r) = r(r-1)+r-\frac{1}{9} = 0.$$

i.e. $(r-\frac{1}{3})(r+\frac{1}{3}) = 0 \Rightarrow r_1 = \frac{1}{3}$, $r_2 = -\frac{1}{3}$.

Setting the series coefficience equal to 0, we have

$$a_n[(n+r)(n+r-1)+(n+r)-\tfrac{1}{9}]+a_{n-2}=0$$

$$\Rightarrow \quad a_n=\frac{-a_{n-2}}{(n+r-\frac{1}{3})(n+r+\frac{1}{3})}, \quad n\geq 2.$$

For $r_1=\frac{1}{3}$, $a_n=\dfrac{-a_{n-2}}{n(n+\frac{2}{3})}$,

Given a_0,
$$\begin{cases} a_2=-\dfrac{3}{16}a_0 \\ a_4=\dfrac{9}{896}a_0 \\ a_6=-\dfrac{9}{35840}a_0 \end{cases}$$

$\ldots\ldots,$

By induction method, $a_{2n}=\dfrac{(-1)^n 3^n a_0}{2^{2n}\cdot n!\cdot \prod_{k=1}^{n}(3k+1)}$, $n\geq 1$.

Given a_1,
$$\begin{cases} a_3=-\dfrac{1}{11}a_1, \\ a_5=\dfrac{3}{935}a_1, \\ a_7=-\dfrac{9}{150535}a_1, \end{cases}$$

$\ldots\ldots$

By induction method, $a_{2n+1}=\dfrac{(-1)^n 3^n 2^n n!\, a_1}{(2n+1)!\prod_{k=1}^{n}(6k+5)}$, $\forall n\geq 1$.

Thus, $y_1=x^{r_1}\left[\sum_{n=0}^{\infty}a_{2n}x^{2n}+\sum_{n=0}^{\infty}a_{2n+1}x^{2n+1}\right]$

$\begin{pmatrix}a_0\\a_1\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix}$

$$\Longrightarrow \begin{cases} y_1=x^{\frac{1}{3}}\Big[1+\sum_{n=1}^{\infty}\dfrac{(-1)^n 3^n}{2^{2n}\cdot n!\prod_{k=1}^{n}(3k+1)}x^{2n}+x+ \\ \qquad \sum_{n=1}^{\infty}\dfrac{(-1)^n 3^n 2^n\cdot n!}{(2n+1)!\prod_{k=1}^{n}(6k+5)}x^{2n+1}\Big], \quad \forall x>0. \end{cases}$$

For $r_2 = -\frac{1}{3}$, $\quad a_n = -\dfrac{a_{n-2}}{n\left(n-\frac{2}{3}\right)}$.

Given a_0,
$$\begin{cases} a_2 = -\frac{3}{8} a_0, \\ a_4 = \frac{9}{320} a_0, \\ a_6 = -\frac{27}{30720} a_0, \end{cases}$$

$\cdots\cdots$

$$a_{2n} = \frac{(-1)^n 3^n a_0}{2^{2n} \cdot n! \prod_{k=1}^{n} (3k-1)}, \quad n \geq 1.$$

Given a_1,
$$\begin{cases} a_3 = -\frac{1}{7} a_1, \\ a_5 = \frac{3}{455} a_1, \\ a_7 = -\frac{9}{60515} a_1, \end{cases}$$

$\cdots\cdots$

$$a_{2n+1} = \frac{(-1)^n 3^n \cdot 2^n \cdot n!\, a_1}{(2n+1)! \prod_{k=1}^{n} (6k+1)}, \quad \forall n \geq 1.$$

Thus, $y_2 = x^{-\frac{1}{3}} \left[\sum_{n=0}^{\infty} a_{2n} x^{2n} + \sum_{n=0}^{\infty} a_{2n+1} x^{2n+1} \right]$.

$\begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Longrightarrow$
$$y_2 = x^{-\frac{1}{3}} \left[2 + \sum_{n=1}^{\infty} \frac{(-1)^n \cdot 3^n}{2^{2n} \cdot n! \prod_{k=1}^{n} (3k-1)} x^{2n} + \sum_{n=1}^{\infty} \frac{(-1)^n \cdot 3^n \cdot 2^n \cdot n!}{(2n+1)! \prod_{k=1}^{n} (6k-1)} x^{2n+1} \right], \quad \forall x > 0.$$

$\blacksquare$

4. $xy'' + y' - y = 0$. $(*)$

Solve. $P(x) = x$, $Q(x) = 1$, $R(x) = -1$. $P(0) = 0$.

$$\lim_{x \to 0} x \cdot \frac{Q}{P} = 1 < \infty \quad \text{and} \quad \lim_{x \to 0} x^2 \frac{R}{P} = 0 < \infty.$$

$\Longrightarrow$ 0 is a regular singular point.

Letting $y = \sum_{n=0}^{\infty} a_n x^{n+r} \Rightarrow y' = \sum_{n=0}^{\infty} a_n (n+1) x^{n+r-1}$

$$y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}, \quad \text{by } (*)$$

$$\Rightarrow \quad a_0 [r(r-1)+r] x^{r-1} + \sum_{n=0}^{\infty} x^{n+r} [a_{n+1}(n+r+1)^2 - a_n] = 0$$

Assume $a_0 \neq 0$, $F(r) = r^2 = 0 \Rightarrow r_1 = r_2 = 0$.

Setting series Coeffience equal to 0, we have

$$a_{n+1} = \frac{a_n}{(n+r+1)^2}, \quad n \geq 0.$$

For $r_1 = 0$, $a_{n+1} = \frac{a_n}{(n+1)^2}$, $n \geq 0$.

Given a_0, $a_1 = a_0$

$a_2 = \frac{1}{4} a_0$,

$a_3 = \frac{1}{4 \cdot 9} a_0 = \frac{1}{36} a_0$.

$a_4 = \frac{1}{4 \cdot 9 \cdot 16} a_0 = \frac{1}{576} a_0$,

...

$$\Rightarrow \quad a_n = \frac{a_0}{\prod_{k=1}^{n} k^2}, \quad \forall n \geq 1.$$

Thus $y_1 = x^0 \left[1 + \sum_{n=1}^{\infty} \frac{1}{\prod_{k=1}^{n} k^2} x^n \right] a_0$.

Letting $a_0 = 1$, we have

$$y_1 = 1 + \sum_{n=1}^{\infty} \frac{1}{(n!)^2} x^n.$$

8. $2x^2 y'' + 3x y' + (2x^2 - 1) y = 0 \quad (*)$

Solve. $P(x) = 2x^2$, $Q(x) = 3x$, $R(x) = 2x^2 - 1$, $P(0) = 0$.

$$\left\{ \lim_{x\to 0} x\frac{Q}{P} = \frac{3}{2}, \quad \lim_{x\to 0} x^2\frac{R}{P} = -\frac{1}{2}. \right.$$

$\Rightarrow$ 0 is regular.

Let $y = \sum_{n=0}^{\infty} a_n x^{n+r} \Rightarrow y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1}$

$$y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}$$

By (*), $a_0[2r(r-1)+3r-1]x^r + a_1[2r(r+1)a_1+3(r+1)-1]x^{r+1}$

$$+ \sum_{n=2}^{\infty} x^{n+r}\left\{a_n[2(n+r)(n+r-1)+3(n+r)-1]+2a_{n-2}\right\} = 0.$$

$\overset{a_n\neq 0}{\Longrightarrow}$ $F(r) = 2r^2+r-1 = 0 \iff r_1 = \frac{1}{2}, r_2 = -1$

Moreover, $a_n = \dfrac{-2a_{n-2}}{[2(n+r)-1](n+r+1)}, \quad \forall n \geq 2.$

For $r_1 = \frac{1}{2}$, $a_n = -\dfrac{2a_{n-2}}{n(2n+3)}$. $n \geq 2$.

Given a_0, $a_2 = -\frac{1}{7}a_0$,

$$a_4 = \frac{1}{154}a_0$$

$$a_6 = -\frac{1}{6930}a_0$$

$\cdots$

$$\Rightarrow a_{2n} = \frac{(-1)^n a_0}{n!\prod_{k=1}^{n}(4k+3)}, \quad n \geq 0.$$

Given a_1, $a_3 = -\frac{2}{27}a_1$,

$$a_5 = \frac{4}{1755}a_1$$

$$a_7 = \frac{-8}{208845}a_1$$

$\cdots$

$$\Rightarrow \quad a_{2n+1} = \frac{(-1)^n \, 2^{2n} \cdot n! \quad a_1}{(2n+1)! \prod_{k=1}^{n} (4k+5)}, \quad \forall\, n \geq 0.$$

$$y_1 = x^{r_1} \left[\sum_{n=0}^{\infty} a_{2n} x^{2n} + \sum_{n=0}^{\infty} a_{2n+1} x^{2n+1} \right]$$

$\begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\Longrightarrow \quad y_1 = x^{\frac{1}{2}} \Big[1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n! \prod_{k=1}^{n} (4k+3)} x^{2n} + x + \sum_{n=1}^{\infty} \frac{(-1)^n \, 2^{2n} \, n!}{(2n+1)! \prod_{k=1}^{n} (4k+5)} x^{2n+1} \Big], \quad \forall\, x > 0.$$

For $r_2 = -1$. $\quad a_n = -\dfrac{2 a_{n-2}}{n(2n+1)}$,

Given a_0, $\quad a_2 = -\frac{1}{5} a_0$,

$a_4 = \frac{1}{90} a_0$,

$a_6 = -\frac{1}{3510} a_0$. $\quad \cdots$

$$\Rightarrow \quad a_{2n} = \frac{(-1)^n \, a_0}{n! \prod_{k=1}^{n} (4k+1)}, \quad \forall\, n \geq 0.$$

Given a_1, $\quad a_3 = \frac{-2}{21} a_1$,

$a_5 = \frac{4}{1155} a_1$,

$a_7 = -\frac{8}{121275} a_1$, $\quad \cdots$,

$$\Rightarrow \quad a_{2n+1} = \frac{(-1)^n \, 2^{2n} \cdot n! \cdot a_1}{(2n+1)! \prod_{k=1}^{n} (4k+3)}, \quad n \geq 0.$$

$$y_2 = x^{r_2} \left[\sum_{n=0}^{\infty} a_{2n} x^{2n} + \sum_{n=0}^{\infty} a_{2n+1} x^{2n+1} \right]$$

$\begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\Longrightarrow \quad y_2 = x^{-1} \Big[1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n! \prod_{k=1}^{n} (4k+1)} x^{2n} + x +$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n} n!}{(2n+1)! \prod_{k=1}^{n}(4k+3)} x^{2n+1} \Big], \quad \forall x>0.$$

§9.6. 2. $x^2y'' - x(2+x)y' + (2+x^2)y = 0.$

Solve. $P(x) = x^2$, $Q(x) = -x(2+x)$, $R(x) = 2+x^2$. $P(0)=0$.

$$\lim_{x\to 0} x \frac{Q}{P} = \lim_{x\to 0}(-2-x) = -2, \text{ and}$$

$$\lim_{x\to 0} x^2 \frac{R}{P} = \lim_{x\to 0}(2+x^2) = 2.$$

$\Rightarrow$ 0 is regular.

$$F(r) = r(r-1) - 2r + 2$$

$$= (r-1)(r-2) = 0 \quad \Rightarrow \quad r_1 = 1, \quad r_2 = 2.$$

6. $2x(x+2)y'' + y' - xy = 0$

Solve. $P(x) = 2x(x+2)$, $Q(x) = 1$, $R(x) = -x$. $P(0) = P(-2) = 0$.

For $x=0$, $\displaystyle\lim_{x\to 0} x \frac{Q(x)}{P(x)} = \frac{1}{4}$,

$$\lim_{x\to 0} x^2 \frac{R(x)}{P(x)} = 0.$$

For $x=-2$, $\displaystyle\lim_{x\to -2}(x+2)\frac{Q(x)}{P(x)} = \lim_{x\to -2}\frac{1}{2x} = -\frac{1}{4}$,

$$\lim_{x\to -2}(x+2)^2 \frac{R(x)}{P(x)} = \lim_{x\to -2}\left[-\frac{(x+2)}{2}\right] = 0.$$

$\Rightarrow$ 0, -2 are regular.

For $x=0$, $F(r) = r(r-1) + \frac{1}{4}r = 0 \quad \Rightarrow r_1 = \frac{3}{4}$, $r_2 = 0$.

For $x=-2$, $F(r)=r(r-1)-\frac{1}{4}r=0 \Rightarrow r_1=\frac{5}{4}, r_2=0.$ ☒

14. $xy''+2xy'+6e^x y=0.$ $(*)$

Solve. $P(x)=x$, $Q(x)=2x$, $R(x)=6e^x$, $P(0)=0.$

$$\lim_{x\to 0} x\frac{Q}{P}=0. \quad \lim_{x\to 0} x^2\frac{R}{P}=\lim_{x\to 0} 6xe^x=0.$$

$\Rightarrow$ 0 is regular.

$F(r)=r(r-1)=0 \Rightarrow r_1=1, r_2=0.$

For $r_1=1$. Let $y=\sum_{n=0}^{\infty} a_n x^{n+r_1}=\sum_{n=0}^{\infty} a_n x^{n+1}.$

$\Rightarrow y'=\sum_{n=0}^{\infty}(n+1)a_n x^n, \quad y''=\sum_{n=1}^{\infty} n(n+1)a_n x^{n-1}.$

By $(*)$ $\Rightarrow$ $\sum_{n=1}^{\infty} n(n+1)a_n x^n+2\sum_{n=1}^{\infty} n a_{n-1}x^n+\underline{6e^x\sum_{n=0}^{\infty} a_n x^{n+1}}=0$ $(**)$

Since $e^x\sum_{n=0}^{\infty} a_n x^{n+1}=\sum_{n=0}^{\infty}\frac{x^n}{n!}\sum_{n=0}^{\infty} a_n x^{n+1}$

$$=(a_0x+a_1x^2+\cdots)+(a_0x^2+a_1x^3+\cdots)+\frac{1}{2}(a_0x^3+a_1x^4+\cdots)+\cdots,$$

it follows from $(**)$ that

$$\sum_{n=1}^{\infty} n(n+1)a_n x^n+2\sum_{n=1}^{\infty} n a_{n-1}x^n+6(a_0x+a_1x^2+\cdots)+6(a_0x^2+a_1x^3+\cdots)+3(a_0x^3+a_1x^4+\cdots)+\cdots=0,$$

$$\Rightarrow \begin{cases} x(2a_1+2a_0+6a_0)=0 \\ x^2(6a_2+4a_1+6a_1+6a_0)=0 \\ x^3(12a_3+6a_2+6a_2+6a_1+3a_0)=0 \\ \cdots \end{cases}$$

Let $a_0 = 1$

$$\Rightarrow \begin{cases} a_1 = -4 \\ a_2 = \frac{17}{3} \\ a_3 = -\frac{47}{12} \quad \cdots \end{cases}$$

$$\Rightarrow y_1(x) = |x| - 4|x|^2 + \frac{17}{3}|x|^3 - \frac{47}{12}|x|^4 + \cdots$$

For $r_2 = 0$, $y_2 = a y_1(x) \ln(x) + \left(1 + \sum_{n=1}^{\infty} c_n x^n\right) x^0$

$$y_2' = a y_1' \ln|x| + \frac{a}{x} y_1 + \sum_{n=1}^{\infty} n c_n x^{n-1}$$

$$y_2'' = a y_1'' \ln(x) + \frac{2ay_1^2}{x} - \frac{ay_1}{x^2} + \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$$

By (*) $\Rightarrow$

$$a \ln(x) \underline{[y_1'' + 2x \overset{0}{\overset{\shortparallel}{y_1'}} + b e^x y_1]} + 2ay_1' - \frac{a}{x} y_1 + 2ay_1$$

$$+ \sum_{n=2}^{\infty} n(n-1) c_n x^{n-1} + 2\sum_{n=1}^{\infty} n c_n x^n + b e^x \sum_{n=0}^{\infty} c_n x^n = 0.$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1) c_n x^{n-1} + 2\sum_{n=1}^{\infty} n c_n x^n + b e^x \sum_{n=1}^{\infty} c_n x^n + b e^x$$

$$= \frac{a}{x} y_1 - 2ay_1' - 2ay_1.$$

$\underline{\underline{e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}}} \Rightarrow$ By a similar argument to above part, letting

$c_1 = 0$, we have $\begin{cases} a = -6 \\ 2c_2 + 8c_1 + 6 = 10a \\ 6c_3 + 10c_2 + 6c_1 + 3 = -\frac{61}{3} a \\ 12c_4 + 12c_3 + 6c_2 + 3c_1 + 1 = \frac{193}{12} a \\ \cdots \end{cases}$

$$\Rightarrow y_2(x) = -6 y_1 \ln|x| + \left[1 - 33|x|^2 + \frac{409}{6}|x|^3 - \frac{1595}{24}|x|^4 + \cdots\right]$$

$\boxtimes$

16. $xy'' + y = 0 \quad (*)$

Solve. $P(x) = x$, $Q(x) = 0$, $R(x) = 1$. $\quad P(0) = 0$.

$$\lim_{x\to 0} x \frac{Q}{P} = 0, \quad \lim_{x\to 0} x^2 \frac{R}{P} = \lim_{x\to 0} x = 0.$$

$\Rightarrow$ 0 is regular.

$\therefore F(r) = r(r-1) = 0$, $\quad r_1 = 1$, $r_2 = 0$.

For $r_1 = 1$ $\quad y = \sum_{n=0}^{\infty} a_n x^{n+1} \Rightarrow y' = \sum_{n=0}^{\infty} a_n (n+1) x^n$

$$y'' = \sum_{n=1}^{\infty} n(n+1) a_n x^{n-1}.$$

By $(*)$ $\Rightarrow$ $\sum_{n=1}^{\infty} x^n [n(n+1) a_n + a_{n-1}] = 0$

$\Rightarrow \quad a_n = -\frac{a_{n-1}}{n(n+1)}, \quad n \geq 1.$

Letting $a_0 = 1$, we have

$$a_1 = -\frac{1}{2}, \quad a_2 = \frac{1}{12}, \quad a_3 = -\frac{1}{144}, \ldots$$

$\Rightarrow \quad y_1 = |x| - \frac{1}{2}|x|^2 + \frac{1}{12}|x|^3 - \frac{1}{144}|x|^4 + \cdots$

$$= |x| + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!\,(n+1)!} x^n.$$

For $r_2 = 0$, let $y_2 = a y_1(x) \ln(x) + [1 + \sum_{n=1}^{\infty} c_n x^n]$.

$y_2' = a y_1' \ln(x) + \frac{a y_1}{x} + \sum_{n=1}^{\infty} n c_n x^{n-1}$.

$y_2'' = a y_1'' \ln(x) + \frac{a}{x} y_1' - \frac{a y_1}{x^2} + \frac{a y_1'}{x} +$

$$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}.$$

$\Rightarrow 1 + \sum_{n=1}^{\infty} c_n x^n + \sum_{n=2}^{\infty} n(n-1) c_n x^{n-1} = \frac{a}{x} y_1 - 2a y_1'$

$$= -a + \frac{3}{2}ax - \frac{5}{12}x^2 a + \frac{7}{144}ax^3 + \cdots$$

Setting coefficients of x^n, $n \geq 0$ equal to each other,

we have $\begin{cases} a = -1 \\ C_2 = -\frac{5}{4} \\ C_3 = \frac{5}{18} \\ C_4 = -\frac{47}{1728}, \quad \cdots \end{cases}$

$$\Rightarrow y_2 = -y_1 \ln|x| + 1 + |x| - \frac{5}{4}|x|^2 + \frac{5}{18}|x|^3 - \frac{47}{1728}|x|^4 + \cdots$$

$\blacksquare$.

§ 9.7 2. $x^2 y'' + 3xy' + (1+x)y = 0$ $(*)$

Solve. $P(x) = x^2$, $Q(x) = 3x$ $R(x) = 1+x$, $P(0) = 0$,

$$\lim_{x \to 0} x \frac{Q}{P} = 3, \quad \lim_{x\to 0} x^2 \frac{R}{P} = 1$$

$\Rightarrow$ 0 is regular.

$$F(r) = r(r-1) + 3r + 1 = 0$$

$$\Rightarrow r_1 = r_2 = -1.$$

For $r_1 = -1$, $y = \sum_{n=0}^{\infty} a_n x^{n-1}$, similar $\cdots$

By $(*)$, $\sum_{n=0}^{\infty} x^n \{[n(n-1) + 3n + 1]a_{n+1} + a_n\} = 0$.

$$\Rightarrow a_{n+1} = -\frac{a_n}{(n+1)^2}.$$

Given a_0, $a_1 = -a_0$

$a_2 = \frac{a_0}{4}$

$a_3 = -\frac{a_0}{36}$

$a_4 = \frac{a_0}{576}$

$$\Rightarrow \quad a_n = \frac{(-1)^n a_0}{(n!)^2}, \quad n \geq 0.$$

Taking $a_0 = 1$, we have

$$y_1(x) = x^{-1} + \sum_{n=1}^{\infty} \frac{(-1)^n}{(n!)^2} x^{n-1}, \quad \forall x > 0.$$

Since $r_1 = r_2$, $\quad y_2(x) = y_1 \ln(x) + x^{-1} \sum_{n=1}^{\infty} b_n x^n \quad (**)$

where $b_n = a_n'$

$\boxed{\begin{array}{l} \text{If } f(x) = (x-\alpha_1)^{\beta_1}(x-\alpha_2)^{\beta_2} \cdots (x-\alpha_n)^{\beta_n}, \text{ and} \\ x \neq \alpha_n \quad \text{then } \dfrac{f'(x)}{f(x)} = \dfrac{\beta_1}{x-\alpha_1} + \dfrac{\beta_2}{x-\alpha_2} + \cdots + \dfrac{\beta_n}{x-\alpha_n}. \quad \text{Pg-II} \end{array}}$

Taking $\beta_1 = \cdots = \beta_n = -2$ and $x = r = -1 \neq \alpha_n$,

we see that $\dfrac{a_n'(-1)}{a_n(-1)} = -2\left(1 + \dfrac{1}{4} + \dfrac{1}{9} + \cdots\right)$

$$\Rightarrow b_n(-1) = a_n'(-1) = -2\left[1 + \frac{1}{4} + \frac{1}{9} + \cdots\right] \frac{(-1)^n a_0}{(n!)^2}$$

By $(**)$, letting $a_0 = 1$, we have

$$y_2(x) = y_1(x)\ln(x) + 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1} S_n}{(n!)^2} x^{n-1},$$ where

$$S_n = \sum_{k=1}^{n} \frac{1}{k^2}.$$

$\square$