HW-1.

Section 9.2 : Q4. Q6. Q8.

Section 9.3 : Q6. Q8.

Section 9.4 : Q6, Q10, Q30.

Section 9.2.

(15). Q4 : Note that all coefficients are constant, so each point is ordinary point.

Assume $y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n$, then we have

$$y' = \sum_{n=1}^{\infty} n \cdot a_n \cdot (x-1)^{n-1} = \sum_{n=0}^{\infty} (n+1) \cdot a_{n+1} \cdot (x-1)^n$$

$$y'' = \sum_{n=2}^{\infty} n \cdot (n-1) \cdot a_n \cdot (x-1)^{n-2} = \sum_{n=0}^{\infty} (n+2) \cdot (n+1) \cdot a_{n+2} \, (x-1)^n$$

substituting into the equation yields:

$$\sum_{n=0}^{\infty} \left[9(n+2) \cdot (n+1) \cdot a_{n+2} + 6(n+1) \cdot a_{n+1} + a_n \right] \cdot (x-1)^n = 0.$$

Then we get the following recurrence relation :

$$9(n+2) \cdot (n+1) \cdot a_{n+2} + 6(n+1) \cdot a_{n+1} + a_n = 0 , \quad n = 0, 1, 2, \cdots$$

That is,

$$a_{n+2} = -\frac{6(n+1) \cdot a_{n+1} + a_n}{9(n+2)(n+1)} , \quad n = 0, 1, 2, \cdots$$

(1) Take $a_0 = 1$, $a_1 = 0$. we have :

$n=0$: $a_2 = -\frac{(6a_1 + a_0)}{18} = -\frac{1}{18}$

$n=1$: $a_3 = -\frac{(12a_2 + a_1)}{54} = -\frac{12(-\frac{1}{18})}{54} = \frac{1}{81}$

$n=2$: $a_4 = -\frac{(18a_3 + a_2)}{108} = -\frac{18 \cdot \frac{1}{81} - \frac{1}{18}}{108} = -\frac{1}{648}$

Thus, the first 4 terms are :

$$y_1(x) = a_0 + a_1(x-1) + a_2(x-1)^2 + a_3(x-1)^3 + \cdots$$

$$= 1 - \frac{1}{18}(x-1)^2 + \frac{1}{81}(x-1)^3 - \frac{1}{648}(x-1)^4 + \cdots$$

(2) Take $a_0 = 0$ and $a_1 = 1$.

$n=0$: $a_2 = -\frac{(6a_1 + a_0)}{18} = -\frac{1}{3}$

$n=1$: $a_3 = -\frac{(12a_2 + a_1)}{54} = -\frac{12 \times (-\frac{1}{3}) + 1}{54} = \frac{1}{18}$

$n=2$: $a_4 = -\frac{(18a_3 + a_2)}{108} = -\frac{18 \cdot \frac{1}{18} - \frac{1}{3}}{108} = -\frac{1}{162}$

Thus, $y_2(x) = (x-1) - \frac{1}{3}(x-1)^2 + \frac{1}{18}(x-1)^3 - \frac{1}{162}(x-1)^4 + \cdots$

(15) Q6: Assume $y(x) = \sum_{n=0}^{\infty} a_n \cdot (x-1)^n$, then we have.

$$y' = \sum_{n=1}^{\infty} n \cdot a_n \cdot (x-1)^{n-1} = \sum_{n=0}^{\infty} (n+1) \cdot a_{n+1} \cdot (x-1)^n$$

$$y'' = \sum_{n=2}^{\infty} n \cdot (n-1) \cdot a_n (x-1)^{n-2} = \sum_{n=0}^{\infty} (n+2) \cdot (n+1) \cdot a_{n+2} \cdot (x-1)^n$$

Since $y'' + x \cdot y' = y'' + (x-1) \cdot y' + y' = 0$

substituting into the equation yields:

$$\sum_{n=0}^{\infty} [(n+2) \cdot (n+1) \cdot a_{n+2} + (n+1) \cdot a_{n+1}] \cdot (x-1)^n + \sum_{n=0}^{\infty} (n+1) \cdot a_{n+1} \cdot (x-1)^{n+1} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} [(n+2) \cdot (n+1) \cdot a_{n+2} + (n+1) \cdot a_{n+1}] (x-1)^n + \sum_{n=1}^{\infty} n \cdot a_n \cdot (x-1)^n = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} [(n+2) \cdot (n+1) \cdot a_{n+2} + (n+1) \cdot a_{n+1}] + \sum_{n=1}^{\infty} n \cdot a_n = 0$$

For $n=0$, we have : $2a_2 + a_1 = 0 \Rightarrow a_2 = -\frac{1}{2} a_1$

For $n \geq 1$, we get the recurrence relation:

$$(n+2) \cdot (n+1) \cdot a_{n+2} + (n+1) \cdot a_{n+1} + n \cdot a_n = 0$$

That is,

$$a_{n+2} = -\frac{(n+1) \cdot a_{n+1} + n a_n}{(n+2) \cdot (n+1)}, \quad n = 1, 2, \cdots$$

(1) Take $a_0 = 1$, $a_1 = 0$.

$n=0$: $a_2 = -\frac{a_1}{2} = 0$

$n=1$: $a_3 = -\frac{0}{6} = 0$

...

Thus, we have $y_1(x) \equiv 1$.

(2) Take $a_0 = 0$ and $a_1 = 1$

$n=0$: $a_2 = -\frac{1}{2}$

$n=1$: $a_3 = -\frac{1-1}{6} = 0$

$n=2$: $a_4 = -\frac{-1}{12} = \frac{1}{12}$

Thus, we have $y_2(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{12}(x-1)^4 + \cdots$

(15) Q18: Assume $y(x) = \sum_{n=0}^{\infty} a_n \cdot x^n$, then we have.

$$y' = \sum_{n=1}^{\infty} n \cdot a_n \cdot x^{n-1} = \sum_{n=0}^{\infty} (n+1) \cdot a_{n+1} \cdot x^{n-2}$$

$$y'' = \sum_{n=2}^{\infty} n \cdot (n-1) \cdot a_n \, x^{n-2} = \sum_{n=0}^{\infty} (n+2) \cdot (n+1) \cdot a_{n+2} \cdot x^n.$$

substituting into the equation yields:

$$(1-x) \cdot \sum_{n=0}^{\infty} (n+2) \cdot (n+1) \cdot a_{n+2} \cdot x^n + x \cdot \sum_{n=0}^{\infty} (n+1) \cdot a_{n+1} \cdot x^n - \sum_{n=0}^{\infty} a_n \cdot x^n = 0.$$

$$\Rightarrow \sum_{n=0}^{\infty} \left[(n+2) \cdot (n+1) \cdot a_{n+2} - a_n \right] \cdot x^n + \sum_{n=0}^{\infty} \left[(n+1) \cdot a_{n+1} - (n+2) \cdot (n+1) \cdot a_{n+2} \right] \cdot x^{n+1} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} \left[(n+2) \cdot (n+1) \cdot a_{n+2} - a_n \right] \cdot x^n + \sum_{n=1}^{\infty} \left[n \cdot a_n - (n+1) \cdot n \cdot a_{n+1} \right] \cdot x^n = 0$$

For $n=0$, we have: $2a_2 - a_0 = 0 \Rightarrow a_2 = \frac{1}{2} a_0$

For $n \geq 1$, we get the recurrence relation:

$$(n+2) \cdot (n+1) \cdot a_{n+2} - n(n+1) \cdot a_{n+1} + (n-1) a_n = 0$$

That is,

$$a_{n+2} = \frac{n \cdot (n+1) \cdot a_{n+1} - (n-1) a_n}{(n+2) \cdot (n+1)}, \quad n = 1, 2, \cdots$$

(1) Take $a_0 = 1$, $a_1 = 0$.

$n=0$: $a_2 = \frac{a_0}{2} = \frac{1}{2}$

$n=1$: $a_3 = \frac{2a_2 - 0}{6} = \frac{1}{6}$

$n=2$: $a_4 = \frac{6a_3 - a_2}{12} = \frac{1}{24}$

...

Thus. we have $y_1(x) = 1 + 0 \cdot x + \frac{1}{2} x^2 + \frac{1}{6} x^3 + \frac{1}{24} x^4 = \sum_{n=0}^{\infty} \frac{x^n}{n!} - x = e^x - x$

(2) Take $a_0 = 0$ and $a_1 = 1$

$n=0$: $a_2 = 0$

$n=1$: $a_3 = \frac{2a_2 - 0}{6} = 0$

$n=2$: $a_4 = \frac{6a_3 - a_2}{12} = 0$

...

Thus, we have $y_2(x) = x$.

Section 9.3.

(15) Q6: $\because P(x) = x^2-2x-3$, $Q(x) = x$ $R(x) = 4$.

$\therefore p(x) = \frac{x}{x^2-2x-3}$, $q(x) = \frac{4}{x^2-2x-3}$

Let $P(x)=0 \Rightarrow x^2-2x-3 = (x-3)\cdot(x+1) = 0$

$\Rightarrow x_1 = 3$, $x_2 = 1$

$\therefore$ The only singular points are $x=3$ and $x=-1$.

By Theorem 9.3.1 and since P and Q are polynomials, the radius of convergence of $\frac{Q}{P}$ around x_0 is the nearest zero points of P in the complex plane.

For $x_0 = 4$:

$\because |4+1| = 5$, $|4-3| = 1$.

$\therefore$ The radius of convergence $R \geq \min\{1, 5\} = 1$.

For $x_0 = -4$:

$\because |-4+1| = 3$, $|-4-3| = 7$

$\therefore$ The radius of convergence $R \geq \min\{3, 7\} = 3$.

For $x_0 = 0$:

$\because |0+1| = 1$, $|0-3| = 3$

$\therefore$ The radius of convergence $R \geq \min\{1, 3\} = 1$

(10) Q8:

$\because P(x) = x$ $Q(x) = 0$, $R(x) = 1$

$\therefore p(x) = \frac{0}{x} = 0$, $q(x) = \frac{1}{x}$

Let $P(x) = 0 \Rightarrow x = 0$

$\therefore$ The only singular point is $x=0$

For $x_0 = 1$:

$\because |1-0| = 1$

$\therefore$ The radius of convergence $R \geq 1$

Section 9.4.

(10) Q6:

Let $t = x-1$, then we have:

$$t^2 \cdot y'' + 8t \cdot y' + 12y = 0$$

This is the standar form of Cauchy-Euler equation.

Let $y = t^r$, then we have

$$y' = r \cdot t^{r-1}$$

$$y'' = r \cdot (r-1) \cdot t^{r-2}$$

substituting into the equation yields.

$$[r \cdot (r-1) + 8r + 12] t^r = 0$$

$$\Rightarrow r(r-1) + 8r + 12 = 0$$

$$\Rightarrow r^2 + 7r + 12 = 0$$

$$\Rightarrow (r+3)(r+4) = 0$$

$\therefore$ $r_1 = -3$, $r_2 = -4$ $\quad r_1 \neq r_2$

$\therefore$ the general solution is: $y(t) = C_1 \cdot t^{-3} + C_2 \cdot t^{-4}$

where C_1 and C_2 are constants.

$$\therefore \quad y(x) = C_1 \cdot (x-1)^{-3} + C_2 (x-1)^{-4}, \quad x \neq 1.$$

(10) Q10. Let $t = x-2$. then we have

$$t^2 \cdot y'' + 5t y' + 8y = 0$$

Let $y = t^r$, then we have

$$y' = r \cdot t^{r-1}$$

$$y'' = r \cdot (r-1) \cdot t^{r-2}$$

substituting into the equation yields.

$$[r \cdot (r-1) + 5r + 8] t^r = 0$$

$$\Rightarrow r^2 + 4r + 8 = 0 \quad \Rightarrow r_1 = -2+2i, \quad r_2 = -2-2i$$

$\therefore$ the general solution is: $y(t) = [C_1 \cdot \cos(2\ln|t|) + C_2 \cdot \sin(2\ln|t|)] \cdot t^{-2}$

where C_1 and C_2 are constants.

$$\therefore \quad y(x) = [C_1 \cdot \cos(2\ln|x-2|) + C_2 \cdot \sin(2\ln|x-2|)] \cdot (x-2)^{-2}.$$

(10) Q 30.

$P(x) = x^2, \quad Q(x) = 2\cdot(e^x - 1), \quad R(x) = e^{-x}\cdot \cos x.$

Let $P(x) = 0 \Rightarrow x = 0.$

The only singular point is $x = 0.$

$p(x) = \frac{2(e^x-1)}{x^2}, \quad q(x) = \frac{e^{-x}\cdot \cos x}{x^2}$

For $x = 0.$

$\because e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$

$\therefore x\cdot p(x) = \frac{2(e^x-1)}{x} = 1 + \frac{x}{2} + \frac{x^2}{6} + \cdots$

This is a power series center at $x=0.$

Thus, it is analytic at $x = 0.$

obviously, $x^2\cdot q(x) = e^{-x}\cdot \cos x$ is analytic at $x = 0$

$\therefore$ $x=0$ is a regular singular point.