Review for Chapters 9 and 10.

Sections 9.2 and 9.3 Series Solutions Near an Ordinary Point, Parts I and II

- The real number x_0 is an **ordinary point** of the linear equation
$$y'' + p(x)y' + q(x)y = 0$$
if p and q are analytic at x_0.
- If x_0 is an ordinary point of
$$y'' + p(x)y' + q(x)y = 0,$$
then the general solution has a representation of the form
$$y = \sum_{n=0}^{\infty} a_n(x - x_0)^n = a_0 y_1(x) + a_1 y_2(x),$$
where a_0 and a_1 are arbitrary, and y_1 and y_2 are linearly independent series solutions that are analytic at x_0.
- The series solutions y_1 and y_2 can be found by formally substituting $y = \sum_{n=0}^{\infty} a_n(x - x_0)^n$ into $y'' + p(x)y' + q(x)y = 0$, grouping coefficients of like powers of $x - x_0$, and setting each coefficient in the resulting series equal to zero. This yields a **recurrence formula** that, in general, gives a_{n+2} in terms of $a_0, a_1, \ldots, a_{n+1}$ for each $n \geq 0$. Starting with arbitrary a_0 and a_1, we can compute, in order, a_2, a_3, a_4, and so on. Choosing $a_0 = 1$ and $a_1 = 0$ then gives y_1, while choosing $a_0 = 0$ and $a_1 = 1$ gives y_2. In some cases, a general formula can be found for the a_n.
- The radius of convergence for each of the series solutions y_1 and y_2 is greater than, or equal to, the minimum of the radii of convergence of the series for p and q.

Section 9.4 Regular Singular Points

- The real number x_0 is a **singular point** of
$$y'' + p(x)y' + q(x)y = 0$$
if either p or q fails to be analytic at x_0.
- If x_0 is a singular point of $y'' + p(x)y' + q(x)y = 0$ for which $(x - x_0)p(x)$ and $(x - x_0)^2 q(x)$ are both analytic at x_0, then x_0 is a **regular singular point**; otherwise, x_0 is an **irregular singular point**.
- The **Cauchy–Euler equation**
$$x^2 y'' + \alpha x y' + \beta y = 0$$
has a regular singular point at $x = 0$. If the roots of the **indicial equation** $r(r - 1) + \alpha r + \beta = 0$ are r_1 and r_2, then general solutions of the Cauchy–Euler equation on either the interval $x < 0$, or the interval $x > 0$, are

$y = c_1 \lvert x\rvert^{r_1} + c_2 \lvert x\rvert^{r_2}$,	if r_1 and r_2 are real and not equal,
$y = (c_1 + c_2 \ln \lvert x\rvert)\lvert x\rvert^{r_1}$,	if r_1 and r_2 are real and equal,
$y = \lvert x\rvert^{\mu}[c_1 \cos(\nu \ln \lvert x\rvert) + c_2 \sin(\nu \ln \lvert x\rvert)]$,	if $r_1, r_2 = \mu \pm i\nu$ and $\nu \neq 0$.

Sections 9.5 and 9.6 Series Solutions Near a Regular Singular Point, Parts I and II

- If $x = 0$ is a singular point of the differential equation and each of the power series $xp(x) = \sum_{n=0}^{\infty} p_n x^n$ and $x^2 q(x) = \sum_{n=0}^{\infty} q_n x^n$ converges for $|x| < \rho$, where $\rho > 0$, then $x = 0$ is a regular singular point of
$$x^2 y'' + x[xp(x)]y' + [x^2 q(x)]y = 0.$$

▶ The **method of Frobenius.** Substituting a series of the form

$$y = \phi(r, x) = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{r+n}$$

into the equation yields the **indicial equation**

$$F(r) = r(r-1) + p_0 r + q_0 = 0,$$

and a **recurrence relation**

$$F(r+n)a_n + \sum_{k=0}^{n-1} a_k[(r+k)p_{n-k} + q_{n-k}] = 0, \qquad n \geq 1.$$

If the roots r_1 and r_2 of $F(r) = 0$ are real, with $r_1 \geq r_2$, then the general solution on either $-\rho < x < 0$ or $0 < x < \rho$ is

$$y = a_1 y_1(x) + a_2 y_2(x),$$

where

$$y_1(x) = |x|^{r_1}\left[1 + \sum_{n=1}^{\infty} a_n(r_1)x^n\right]$$

and

$$y_2(x) = \begin{cases} |x|^{r_2}[1 + \sum_{n=1}^{\infty} a_n(r_2)x^n], & \text{if } r_1 - r_2 \text{ is not zero or a positive integer,} \\ y_1(x)\ln|x| + |x|^{r_1}\sum_{n=1}^{\infty} b_n(r_1)x^n, & \text{if } r_1 = r_2, \\ ay_1(x)\ln|x| + |x|^{r_2}\sum_{n=1}^{\infty} c_n(r_2)x^n, & \text{if } r_1 - r_2 = N, \text{ a positive integer.} \end{cases}$$

All the coefficients can be determined by substituting the form of the series solution into $x^2y'' + x[xp(x)]y' + [x^2q(x)]y = 0$; the a_n will necessarily satisfy the recurrence relation $F(r+n)a_n + \sum_{k=0}^{n-1} a_k[(r+k)p_{n-k} + q_{n-k}] = 0$. All the power series appearing in the solutions converge at least for $|x| < \rho$.

Section 9.7 Bessel's Equation

▶ **Bessel's equation of order** ν,

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0,$$

has a regular singular point at $x = 0$.

▶ The general solution of **Bessel's equation of order zero** is $y = c_1J_0(x) + c_2Y_0(x)$, $x > 0$, where

$$J_0(x) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m}(m!)^2}$$

and

$$Y_0(x) = \frac{2}{\pi}\left[\left(\gamma + \ln\frac{x}{2}\right)J_0(x) + \sum_{m=1}^{\infty} \frac{(-1)^{m+1}H_m}{2^{2m}(m!)^2}x^{2m}\right].$$

▶ The general solution of **Bessel's equation of order one-half** is $y = c_1J_{1/2}(x) + c_2J_{-1/2}(x)$, $x > 0$, where

$$J_{1/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2}\cos x \qquad \text{and} \qquad J_{-1/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2}\sin x.$$

▶ The general solution of **Bessel's equation of order one** is $y = c_1J_1(x) + c_2Y_1(x)$, $x > 0$, where

$$J_1(x) = \frac{x}{2}\sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m}(m+1)!m!} \qquad \text{and} \qquad Y_1(x) = \frac{2}{\pi}[-y_2(x) + (\gamma - \ln 2)J_1(x)].$$

Section 10.3 Elementary Two-Point Boundary Value Problems

- **The two-point boundary value problem**

  $$y'' + p(x)y' + q(x)y = g(x), \quad \alpha < x < \beta, \qquad y(\alpha) = y_1, \quad y(\beta) = y_2$$

  is **nonhomogeneous** if at least one of the **boundary values** y_1 and y_2 is nonzero, or if the function g is not everywhere zero. If both of the boundary values y_1 and y_2 are zero, and g is the zero function, then the resulting two-point boundary value problem

  $$y'' + p(x)y' + q(x)y = 0, \quad \alpha < x < \beta, \qquad y(\alpha) = 0, \quad y(\beta) = 0$$

  is **homogeneous**.

  - If $y = 0$ is the unique solution of the homogeneous two-point boundary value problem, then the nonhomogeneous two-point boundary value problem has a unique solution.
  - If the solution $y = 0$ of the homogeneous two-point boundary value problem is not unique, then the nonhomogeneous two-point boundary value problem has either no solution or infinitely many solutions.

- The problem

  $$y'' + \lambda y = 0, \quad \alpha < x < \beta, \qquad y(\alpha) = 0, \quad y(\beta) = 0$$

  containing the parameter λ is an example of a special type of two-point boundary value problem, called an **eigenvalue problem**. Values of λ for which nonzero solutions exist are called **eigenvalues**, and the corresponding nonzero solutions are called **eigenfunctions**.

Section 10.4 General Sturm-Liouville Boundary Value Problems

- The solution of many partial differential equations involves the solution of an eigenvalue problem that lies in the class of **regular Sturm–Liouville boundary value problems**:

  $$[p(x)y']' - q(x)y + \lambda r(x)y = 0, \qquad 0 < x < 1,$$
  $$0_1 y(0) + 0_2 y'(0) = 0, \qquad 1_1 y(1) + 1_2 y'(1) = 0,$$

  where the functions p, p', q, and r are continuous on the interval $0 \le x \le 1$ and where $p(x) > 0$ and $r(x) > 0$ at all points in $0 \le x \le 1$.

- Properties of the eigenvalues and eigenvectors of a regular Sturm–Liouville problem are:
  - The eigenvalues and eigenfunctions are real.
  - Eigenfunctions belonging to distinct eigenvalues are orthogonal with respect to the weighted inner product defined by

$$\langle u, v\rangle_r = \int_0^1 r(x)u(x)v(x)\,dx = 0.$$

  - The eigenvalues form an infinite sequence and can be ordered according to increasing magnitude so that

$$\lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_n < \cdots$$

    Moreover $\lambda_n \to \infty$ as $n \to \infty$.
  - The eigenvalues are simple, that is, there is only one independent eigenfunction associated with each eigenvalue.

Section 10.5 Generalized Fourier Series and Eigenfunction Expansions

- If f and f' belong to $PC[\alpha, \beta]$, and $\phi_1, \phi_2, \ldots$ are the eigenfunctions of a regular Sturm–Liouville problem, then the **generalized Fourier series expansion** is given by

$$f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x),$$

$$\varphi'_n = \frac{\varphi_n}{\|\varphi_n\|_r} = \frac{\varphi_n}{\langle \varphi_n, \varphi_n\rangle_r^{\frac{1}{2}}}$$

where the coefficients c_n are defined as

$$c_n = \frac{\langle f, \phi_n\rangle_r}{\langle \phi_n, \phi_n\rangle_r^{\frac{1}{2}}} = \frac{\int_\alpha^\beta r(x)f(x)\phi_n(x)\,dx}{\left(\int_\alpha^\beta r(x)\phi_n(x)\phi_n(x)\,dx\right)^{\frac{1}{2}}}, \quad \text{for } n = 1, 2, \ldots.$$

$$c_n = \langle f, \varphi'_n\rangle_r$$

- If λ_n and ϕ_n are the eigenvalues and eigenvectors, respectively, of the Sturm–Liouville problem

$$L[y] = -[p(x)y']' + q(x)y = \lambda r(x)y,$$
$$\alpha_1 y(0) + \alpha_2 y'(0) = 0, \qquad \beta_1 y(1) + \beta_2 y'(1) = 0,$$

then the associated nonhomogeneous boundary value problem

$$L[y] = -[p(x)y']' + q(x)y = \mu r(x)y + f(x),$$
$$\alpha_1 y(0) + \alpha_2 y'(0) = 0, \qquad \beta_1 y(1) + \beta_2 y'(1) = 0,$$

may be solved by assuming a solution of the form $y = \phi(x) = \sum_{n=1}^{\infty} b_n\phi_n(x)$. Substituting this expansion into the nonhomogeneous equation, using the relation $L[\phi_n] = \lambda_n r\phi_n$, and using the orthogonality conditions $\langle \phi_n, \phi_m\rangle_r = 0$ for $n \neq m$ yield the following result:

$$y = \phi(x) = \begin{cases} \sum_{n=1}^{\infty} \frac{c_n}{\lambda_n - \mu}\phi_n(x), & \text{if } \mu \notin \{\lambda_1, \lambda_2, \ldots\}, \\ \sum_{n\neq m} \frac{c_n}{\lambda_n - \mu}\phi_n(x) + c\,\phi_m(x), & \text{if } \mu = \lambda_m \text{ and } \underbrace{\int_0^1 f(x)\phi_m(x)\,dx}_{c_m} = 0. \end{cases}$$

Here φ_n is the normalized eigenfunction

where $c_n = \langle f, \phi_n\rangle \cancel{dx}$ and c is an arbitrary constant.

- If $\mu = \lambda_m$ and $\underbrace{\langle f, \phi_m\rangle}_{c_m} \cancel{dx} \neq 0$, then the nonhomogeneous problem has no solution.

$$\frac{f(x)}{r(x)} = \sum_{n=1}^{\infty} c_n \cdot \varphi_n(x),$$

$$(\lambda_n - \mu)\, b_n - c_n = 0$$

Section 10.6 Singular Sturm–Liouville Boundary Value Problems

The differential operator $L[y] = -[p(x)y']' + q(x)y$ is said to be **formally self-adjoint**. The corresponding differential equation

$$L[y] = -[p(x)y']' + q(x)y = \lambda r(x)y, \qquad 0 < x < 1,$$

is said to be in **self-adjoint form**. If

(i) p, p', q, and r are continuous on the interval $0 \leq x \leq 1$ and
(ii) $p(x) > 0$ and $r(x) > 0$ at all points in $0 \leq x \leq 1$,

except at one or both endpoints of the interval $[0, 1]$, then a boundary value problem for the equation $L[y] = \lambda r(x)y$ is said to be a **singular Sturm–Liouville problem** on $(0, 1)$.

- Conditions for singular Sturm–Liouville problems that commonly occur in applications are:
  - Left endpoint singular, regular boundary condition at the right endpoint:

    $$p(0) = 0, \qquad y'(1) + h_2 y(1) = 0.$$

  - Right endpoint singular, regular boundary condition at the left endpoint:

    $$y'(0) - h_1 y(0) = 0, \qquad p(1) = 0.$$

  - Both endpoints singular: $p(0) = p(1) = 0$ and no boundary condition specified at $x = 0$ or $x = 1$.
- If there is a sufficiently large class of twice continuously differentiable functions on the interval $(0, 1)$, such that

  $$\int_0^1 \{L[u]v - uL[v]\}\, dx = 0$$

  is true, in the sense of an improper integral, for all u and v in the class, then the singular Sturm–Liouville problem is **self-adjoint**. The set E of λ for which there are nonzero solutions of $L[\phi] = -[p(x)\phi']' + q(x)\phi = \lambda r(x)\phi$ in this class is called the **spectrum** of L. The set E consists of only real numbers; it may be a discrete set of points (eigenvalues); it may be a subinterval of the real line (a **continuous spectrum**); or it may be a mixture of discrete eigenvalues and a continuous spectrum.