§5. Series solutions near a <u>regular</u> singular point (II)

Consider $$L[y] = x^2 y'' + x[xp(x)]\,y' + [x^2 q(x)]\,y = 0, \qquad (5.1)$$

where $$xp(x) = \sum_{n=0}^{\infty} p_n x^n, \quad x^2 q(x) = \sum_{n=0}^{\infty} q_n x^n, \quad |x| < \rho. \qquad (5.2)$$

Let $$y = \varphi(r, x) = x^r \cdot \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{r+n} \qquad (5.3)$$

Let $F(r) = r(r-1) + p_0 r + q_0$. It then follows that

ex. $$L[\varphi] = a_0 F(r) x^r + \sum_{n=1}^{\infty} \left\{ F(r+n) a_n + \sum_{k=0}^{n-1} a_k [(r+k) p_{n-k} + q_{n-k}] \right\}.$$

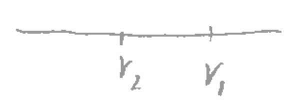
$x^{r+n} = 0$

$\Rightarrow$ $F(r) = 0$ (indicial equation), the roots r_1 and r_2 are called the exponents at the singularity.

and $$\underline{\underline{F(r+n)}} \cdot a_n + \sum_{k=0}^{n-1} a_k [(r+k) p_{n-k} + q_{n-k}] = 0, \quad \forall n \geq 1 \qquad (5.4)$$

(recurrence relation)

<u>Case 1</u>. $F(r) = 0$ has two roots $r_1 \neq r_2$ with $r_1 > r_2$ and $r_1 - r_2$ is not a <u>positive integer</u>



~~Then we have~~ Taking $a_0 = 1$, we then have

For r_1 (since $F(r_1+n)\neq 0$), $y_1(x) = x^{r_1}\left[1+\sum_{n=1}^{\infty} a_n(r_1)\, x^n\right]$, $x>0$ (5.5)

and for r_2 (since $F(r_2+n)\neq 0$), $y_2(x) = x^{r_2}\left[1+\sum_{n=1}^{\infty} a_n(r_2)\cdot x^n\right]$, $x>0$ (5.6)

In the case where $x<0$, we use $|x|^{r_1}$ and $|x|^{r_2}$ instead of x^{r_1} and x^{r_2}, respectively. (why? $\xi = -x = |x|$ when $x<0$)

<u>Example 5.1</u> Find two solutions of $2x(1+x)y''+(3+x)y'-x\cdot y=0$ near the singular point $x_0=0$.

<u>Solution</u>

$$x\cdot p(x) = x\cdot\frac{Q(x)}{P(x)} = x\cdot\frac{3+x}{2x(1+x)} = \frac{3+x}{2\cdot(1+x)} \to \frac{3}{2}, \text{ as } x\to 0$$

and

$$x^2\cdot q(x) = x^2\cdot\frac{R(x)}{P(x)} = x^2\cdot\frac{-x}{2x(1+x)} = \frac{-x^2}{2(1+x)} \to 0, \text{ as } x\to 0.$$

Then $x_0=0$ is a <u>regular</u> singular point. Since $p_0=\frac{3}{2}$, $q_0=0$. The <u>indicial equation</u> is $r(r-1)+\frac{3}{2}r=0$, and hence, $r_1=0$, $r_2=-\frac{1}{2}$, $r_1-r_2=\frac{1}{2}$ (not an integer).

Thus, $y_1(x)=1+\sum_{n=1}^{\infty} a_n(0)\, x^n$, $y_2(x)=|x|^{-\frac{1}{2}}\left[1+\sum_{n=1}^{\infty} a_n(-\tfrac{1}{2})\, x^n\right]$, $0<|x|<\rho$.

For $x\cdot p(x)$, $\rho_1=1$, and for $x^2\cdot q(x)$, $\rho_2=1$. Thus, $\rho\geq 1$.

<u>Case 2</u>. $F(r)$ has roots $r_1=r_2$. Note that $F(r)=(r-r_1)^2$,

$\varphi(x,r)=x^r\cdot\sum_{n=0}^{\infty} a_n(r)\, x^n$, $\left.\frac{\partial\varphi}{\partial r}\right|_{r=r_1}$ is a solution.

$$\Rightarrow y_2(x) = y_1(x)\cdot\ln|x| + |x|^{r_1}\cdot\sum_{n=1}^{\infty} a_n'(r_1)\, x^n \quad (5.7)$$

$b_n(r_1)$

<u>Case 3</u>. $F(r)=0$, $r_1>r_2$, $r_1-r_2=N$ $(N\geq 1)$

$$y_2(x) = a\cdot y_1(x)\ln|x| + |x|^{r_2}\cdot\left[1+\sum^{\infty} c_n(r_1)\, x^n\right]. \quad (5.8)$$

§6. Bessel's Equation

Consider Bessel's equation

$$x^2 \cdot y'' + x y' + (x^2 - \nu^2) y = 0, \qquad (6.1)$$

$$x \cdot \frac{Q(x)}{P(x)} = x \cdot \frac{x}{x^2} = 1, \qquad x^2 \cdot \frac{R(x)}{P(x)} = x^2 \cdot \frac{x^2 - \nu^2}{x^2} = x^2 - \nu^2$$

Then $x_0 = 0$ is a <u>regular</u> singular point, and $p_0 = 1$, $q_0 = -\nu^2$.

Then $F(r) = r(r-1) + p_0 r + q_0 = r(r-1) + r - \nu^2 = r^2 - \nu^2 = 0$

$\Rightarrow$ $r = \pm \nu$. We will consider $\nu = 0$, $\nu = \frac{1}{2}$, $\nu = 1$.

1. $\nu = 0$ (Bessel Equation of order zero)

$$L[y] = x^2 \cdot y'' + xy' + x^2 \cdot y = 0 \qquad (6.2)$$

$\underline{r_1 = r_2 = 0}$

$$y = \varphi(r, x) = a_0 x^r + \sum_{n\geq 1}^{\infty} a_n x^{r+n}$$

$$L[\varphi] = x^2 \sum_{n\geq 0}^{\infty} a_n \underline{(r+n)(r+n-1)} x^{r+n-2} = \sum_{n\geq 0}^{\infty} a_n x^{r+n} \qquad (6.3)$$

$$+ x \cdot \sum_{n\geq 0}^{\infty} a_n \underline{(r+n)} \cdot x^{r+n-1}$$

$$+ x^2 \cdot \sum_{n\geq 0}^{\infty} a_n x^{r+n}$$

$$\Rightarrow \sum_{n\geq 0}^{\infty} a_n \left[(r+n)(r+n-1) + (r+n) \right] x^{r+n} + \sum_{n\geq 0}^{\infty} a_n \cdot x^{\underline{r+n+2}} = 0$$

$n+2 \rightarrow n$

$n \rightarrow n-2$

$$\Rightarrow \sum_{n\geq 0}^{\infty} a_n \left[(r+n)(r+n-1) + (r+n) \right] x^{r+n} + \sum_{n\geq 2}^{\infty} a_{n-2} \cdot x^{r+n} = 0$$

$$\Rightarrow a_0 \left[\underline{r(r-1) + r} \right] x^r + a_1 \cdot \left[\underline{(r+1)r + (r+1)} \right] x^{r+1}$$

$$+ \sum_{n\geq 2}^{\infty} \underline{\left\{ a_n \left[(r+n)(r+n-1) + (r+n) \right] + a_{n-2} \right\}} x^{r+n} = 0$$

$a_0 F(r) = 0$

$\left(\underline{F(r) = r(r-1) + r = 0}, \quad r_1 = r_2 = 0 \right)$,

$a_1 [(r+1) \cdot r + (r+1)] = 0$.

$a_1 \cdot (r+1)^2 = 0$

$$a_n(r) = -\frac{a_{n-2}(r)}{(r+n)(r+n-1) + (r+n)} = -\frac{a_{n-2}(r)}{(r+n)^2}, \quad \forall n \geq 2 \qquad (6.4)$$

For $y_1(x)$, let $r = 0$. $\Rightarrow a_1 = 0$,

$$\xRightarrow{(6.4)} a_3 = a_5 = a_7 = \cdots = 0, \quad \text{and } a_n(0) = -\frac{a_{n-2}(0)}{n^2},$$