§3 Regular singular points

1. Cauchy-Euler equations

Consider $L[y] = x^2 y'' + \alpha x \cdot y' + \beta y = 0, \qquad (3.1)$

when α and β are real constants. Since $P(x) = x^2$, 0 is ~~$P(x) = x^2$~~ the only singular point. Try solution $y = x^r, \quad x > 0 \qquad (3.2)$

(Recall $x^r = e^{r \ln x}$)

$$\Rightarrow L[x^r] = x^2 \cdot r \cdot (r-1) x^{r-2} + \alpha \cdot x \cdot r x^{r-1} + \beta \cdot x^r$$

$$= x^r \left[r(r-1) + \alpha r + \beta \right] = 0$$

$$\Rightarrow F(r) := r(r-1) + \alpha r + \beta = r^2 + (\alpha - 1) r + \beta = 0 \qquad (3.3)$$

Case 1 $F(r) = 0$ has two real roots $r_1 \neq r_2$. Then $y_1(x) = x^{r_1}$ and $y_2(x) = x^{r_2}$ are two solutions of (3.1).

Since $W(x^{r_1}, x^{r_2}) = \begin{vmatrix} x^{r_1} & x^{r_2} \\ r_1 x^{r_1 - 1} & r_2 x^{r_2 - 1} \end{vmatrix} = (r_2 - r_1) x^{r_1 + r_2 - 1} \neq 0$,

it follows that $y = c_1 x^{r_1} + c_2 x^{r_2}, \quad x > 0 \qquad (3.4)$

Case 2 $F(r) = 0$ has one real root $r_1 = r_2$. Let $y_1(x) = x^{r_1}$.

Then $F(r) = (r-r_1)(r-r_2) = (r-r_1)^2$, and hence $F(r_1)=0$, and $F'(r_1)=0$.

Since $L(x^r) = x^r F(r)$, $\Rightarrow \frac{\partial}{\partial r} L[x^r] = \frac{\partial}{\partial r}[x^r \cdot F(r)]$

$\Rightarrow L[\frac{\partial}{\partial r} \cdot x^r] = \underline{x^r \cdot \ln x} \cdot F(r) + x^r \cdot F'(r)$

$\Rightarrow L[x^r \ln x] = x^r \ln x \cdot F(r) + x^r \cdot F'(r)$

Let $r=r_1$, $\Rightarrow L[x^{r_1} \ln x] = 0$, $x>0$

Thus $y_2(x) = x^{r_1} \ln x$ is a solution of (3.1)

Since $W(x^{r_1}, x^{r_1}\ln x) = \begin{vmatrix} x^{r_1} & x^{r_1}\ln x \\ r_1 x^{r_1-1} & \underline{(x^{r_1}\ln x)'} \end{vmatrix} \overset{\text{ex.}}{=\!=} x^{2r_1-1}$,

$\begin{pmatrix} r_1 x^{r_1-1}\ln x \\ + x^{r_1-1} \end{pmatrix}$

$y_1(x)$ and $y_2(x)$ are a fundamental set of solutions for $x>0$, and hence

$$y = c_1 x^{r_1} + c_2 \cdot x^{r_1} \ln x = (c_1 + c_2 \ln x) x^{r_1},$$

$x>0$

<u>Case 3</u> $F(r)=0$ has complex roots $r_1 = u+iv$ and $r_2 = u-iv$, $(v \neq 0)$ (Define $x^r = e^{r\ln x}$) (3.5)

Then x^{u+iv} and x^{u-iv} are two complex solutions of (3.1). Note that $x^{u+iv} = \cancel{x^u \cdot x^{iv} = x^{u}} \; e^{(u+iv)\ln x}$

~~($x^r = e^{r\ln x}$)~~

$$= e^{\mu \cdot \ln x} \cdot e^{i\nu \ln x} \qquad (\text{Euler's formula: } e^{i\alpha} = \cos\alpha + i\sin\alpha)$$

$$= x^{\mu} \cdot \left(\cos(\nu \ln x) + i \sin(\nu \ln x) \right)$$

$$= x^{\mu} \cos(\nu \ln x) + i \cdot x^{\mu} \cdot \sin(\nu \ln x)$$

Thus, $y_1(x) = x^{\mu} \cos(\nu \ln x)$ and $y_2(x) = x^{\mu} \cdot \sin(\nu \ln x)$ are two solutions of (3.1). Since

$$W\left[x^{\mu}\cos(\nu \ln x),\ x^{\mu}\sin(\nu \ln x) \right] \overset{\text{ex.}}{=\!=} \nu \cdot x^{2\mu - 1}, \quad x > 0,$$

these two solutions form a fundamental set of solutions for $x > 0$, and hence,

$$y = c_1 x^{\mu} \cos(\nu \ln x) + c_2 \cdot x^{\mu} \cdot \sin(\nu \ln x), \quad x > 0 \qquad (3.6)$$

Example 3.1 (a) $x^2 y'' + 5xy' + 4y = 0$, $x > 0$

(b) $x^2 y'' + xy' + y = 0$, $x > 0$

Solution (a) $F(r) = r(r-1) + \alpha r + \beta = r(r-1) + 5 \cdot r + 4$
$= r^2 + 4r + 4 = (r+2)^2 = 0$

$\Longrightarrow r_1 = r_2 = -2 \overset{(3.5)}{\Longrightarrow} y = (c_1 + c_2 \ln x)x^{-2}$, $x > 0$.

(b) $F(r) = r(r-1) + 1 \cdot r + 1 = r^2 + 1 = 0 \Longrightarrow r = \pm i$, $\mu = 0$, $\nu = 1$

$\overset{(3.6)}{\Longrightarrow} y = c_1 \cos(\ln x) + c_2 \cdot \sin(\ln x)$, $x > 0$.