§2. Series solutions near an ordinary point (II)

Theorem 2.1 Assume that x_0 is an ordinary point of

$$P(x)y'' + Q(x)y' + R(x)y = 0, \qquad (2.1)$$

that is, $p = \frac{Q}{P}$ and $q = \frac{R}{P}$ are analytic at x_0. Then the following statements are valid:

(i) The general solution of (2.1) is

$$y = \sum_{n=0}^{\infty} a_n (x-x_0)^n = a_0 y_1(x) + a_1 y_2(x), \qquad (2.2)$$

where a_0 and a_1 are arbitrary, and y_1 and y_2 are two power series solutions that are analytic at x_0.

(ii) The solutions y_1 and y_2 form a fundamental set of solutions.

(iii) The radius of convergence for each of y_1 and y_2 is at least as large as the minimum of the radii of convergence of the series for p and q.

Analysis ~~$y = f(x)$~~ Consider $y'' + p(x)y' + q(x)y = 0 \qquad (2.3)$

Assume that $p(x)$ and $q(x)$ has power series expansion for $|x-x_0| <$

$\left(p(x) = \sum_{n\geq 0}^{\infty} p_n (x-x_0)^n, \quad q(x) = \sum_{n\geq 0}^{\infty} q_n (x-x_0)^n \right)$

If $\quad y = \varphi(x) = \sum_{n\geq 0}^{\infty} a_n (x-x_0)^n, \quad y(x_0) = y_0, \quad y'(x_0) = y_1$

then $\quad a_n = \frac{1}{n!} \varphi^{(n)}(x_0) \quad$ (why?) (Taylor series)

Clearly $\quad a_0 = y_0, \quad a_1 = y_1.$

Since $\quad \underline{\varphi''(x) = -p(x)\varphi'(x) - q(x)\varphi(x)} \qquad (2.4)$

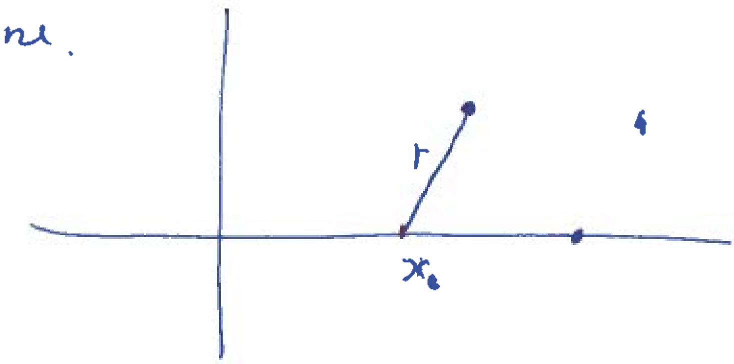
$\Longrightarrow \quad a_2 = \frac{1}{2}\left[-p(x_0)\cdot a_1 - q(x_0)\cdot a_0 \right]$

...... we can ~~compute~~ continue to differentiate (2.4) and determine $a_3, a_4, \ldots$ successively.

<u>Remark 2.1</u> In practice, we substitute $y = \sum_{n\geq 0}^{\infty} a_n (x-x_0)^n$ into (2.1) (or (2.3)) to determine all a_n (as we did in section 1)

<u>Remark 2.1</u> Theorem 2.1 (iii) provides a <u>lower bound</u> on radius of convergence of the series solution.

In general case, we ~~can~~ may use the <u>convergence</u> tests to determine the radii of the power series for p and q. In the case when P, Q, and R are polynomials, then is an easy way. For example, ~~for $p = \frac{Q}{P}$~~, the radius of convergence of the power series for $\frac{Q}{P}$ about x_0 is the distance from x_0 to the nearest zero of P in the complex plane.



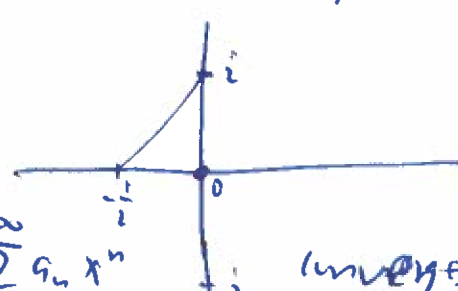
Example 2.1 Determine a lower bound for the radius of convergence of series solutions of

$$(1+x^2)y'' + 2x\cdot y' + 4x^2 y = 0 \qquad (2.5)$$

about $x_0 = 0$ and $x_0 = -\frac{1}{2}$.

Solution. $p(x) = \frac{Q}{P} = \frac{2x}{1+x^2}$ $\quad q(x) = \frac{R}{P} = \frac{4x^2}{1+x^2}$.

$P(x) = 0 \Longrightarrow x = \pm i$



For $x_0 = 0$, $\rho = 1$, $\sum_{n=0}^{\infty} a_n x^n$ converges at least for $|x| < 1$

For $x_0 = -\frac{1}{2}$, $\rho = \sqrt{1 + (\frac{1}{2})^2} = \sqrt{5}/2$, and the series solution $\sum_{n=0}^{\infty} b_n (x+\frac{1}{2})^n$ converges at least for $|x + \frac{1}{2}| < \frac{\sqrt{5}}{2}$

Example 2.2 $y'' + (\sin x) y' + (1+x^2) y = 0$, $\rho = ?$

Solution $p(x) = \sin x$. $q(x) = 1 + x^2$. $r = \infty \Longrightarrow$ the series solution converges for all x