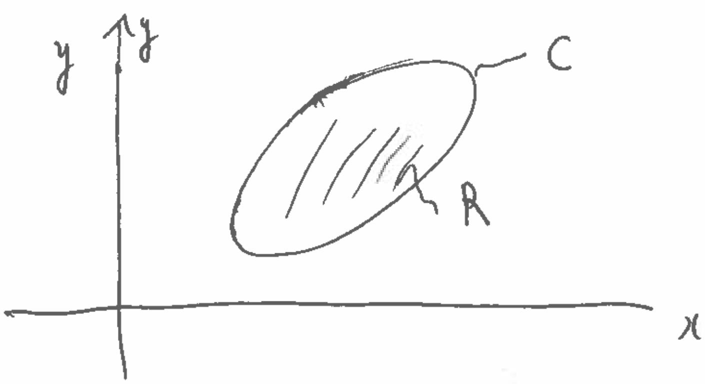
§4. Variational problems for double integrals

The general method of finding necessary conditions for an integral to be stationary can be generalized to multiple integrals.



Consider the integral

$$I(z) = \iint_R f(x, y, z, z_x, z_y)\,dx\,dy, \tag{4.1}$$

where $z = z(x,y)$, $(x,y) \in R$, and the value of $z(x,y)$ on the boundary C is given. Find a stationary function $z = z(x,y)$ that gives a stationary value to the integral $I(z)$.

Assume that $z(x,y)$ is the desired stationary function. Let $\bar{z}(x,y) = z(x,y) + \alpha \cdot \eta(x,y)$, $\alpha \in \mathbb{R}$, where $\eta(x,y) = 0$ for all $(x,y) \in C$. Then the function

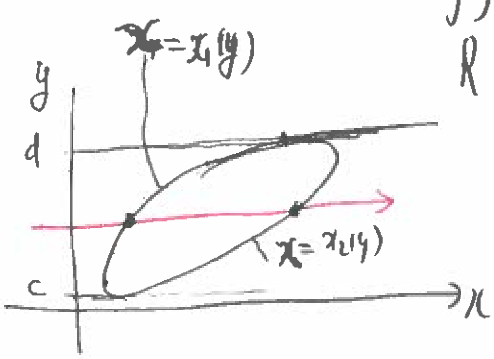
$$I(\alpha) = \iint_R f(x, y, \bar{z}, \bar{z}_x, \bar{z}_y)\, dx\, dy$$

has a stationary value at $\alpha = 0$, and hence, $I'(0) = 0$.

$$\frac{d}{d\alpha} I(\alpha) = \iint_R \left[\frac{\partial f}{\partial z}(x, y, \bar{z}, \bar{z}_x, \bar{z}_y) \cdot \frac{\partial \bar{z}}{\partial \alpha} + \frac{\partial f}{\partial z_x}(x, y, \bar{z}, \bar{z}_x, \bar{z}_y) \cdot \frac{\partial \bar{z}_x}{\partial \alpha} + \frac{\partial f}{\partial z_y}(x, y, \bar{z}, \bar{z}_x, \bar{z}_y) \cdot \frac{\partial \bar{z}_y}{\partial \alpha} \right] dx\, dy$$

$$= \iint_R \left[\frac{\partial f}{\partial z}(x, y, \bar{z}, \bar{z}_x, \bar{z}_y) \cdot \eta(x,y) + \frac{\partial f}{\partial z_x}(x, y, \bar{z}, \bar{z}_x, \bar{z}_y) \cdot \eta_x(x,y) + \frac{\partial f}{\partial z_y}(x, y, \bar{z}, \bar{z}_x, \bar{z}_y) \cdot \eta_y(x,y) \right] dx\, dy.$$

Then
$$I'(0) = \iint_R \left(\frac{\partial f}{\partial z} \cdot \eta + \frac{\partial f}{\partial z_x} \cdot \eta_x + \frac{\partial f}{\partial z_y} \cdot \eta_y \right) dx\, dy = 0 \qquad (4.2)$$

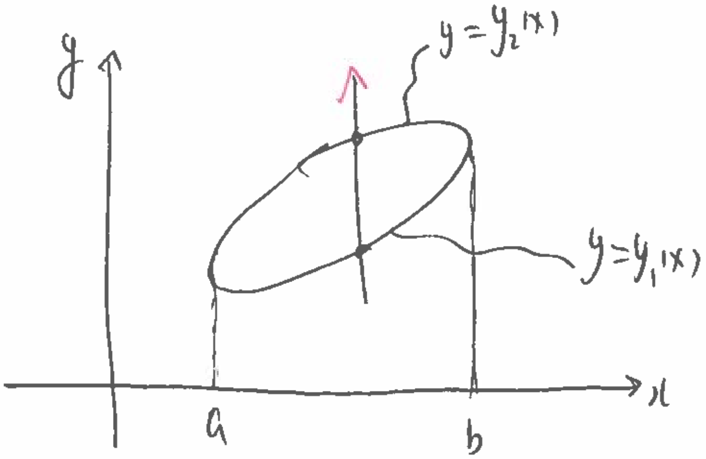


$$\iint_R \frac{\partial f}{\partial z_x} \cdot \eta_x \, dx\, dy = \int_c^d \left(\int_{x_1(y)}^{x_2(y)} \frac{\partial f}{\partial z_x} \cdot \eta_x \, dx \right) dy$$

$$= \int_c^d \left[\left(\frac{\partial f}{\partial z_x} \cdot \eta \right) \Bigg|_{x_1(y)}^{x_2(y)} - \int_{x_1(y)}^{x_2(y)} \eta \cdot \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial z_x} \right) dx \right] dy$$

$$= - \int_c^d \left[\int_{x_1(y)}^{x_2(y)} \eta \cdot \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial z_x} \right) dx \right] dy$$

$$= - \iint_R \eta \cdot \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial z_x} \right) dx\, dy .$$



Similarly, we have

$$\iint_R \frac{\partial f}{\partial z_y} \cdot \eta_y \, dx\, dy$$

$$= \int_a^b \left(\int_{y_1(x)}^{y_2(x)} \frac{\partial f}{\partial z_y} \cdot \eta_y \, dy \right) dx$$

$$= \int_a^b \left[\frac{\partial f}{\partial z_y} \cdot \eta \Bigg|_{y_1(x)}^{y_2(x)} - \int_{y_1(x)}^{y_2(x)} \eta \cdot \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial z_y} \right) dy \right] dx$$

$$= - \int_a^b \left[\int_{y_1(x)}^{y_2(x)} \eta \cdot \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial z_y} \right) dy \right] dx = - \iint_R \eta \cdot \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial z_y} \right) dx\, dy$$

Thus, (4.2) becomes

$$\iint_R \eta \cdot \left[\frac{\partial f}{\partial z} - \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial z_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z_y}\right) \right] dx\,dy = 0. \quad (4.3)$$

Since $\eta(x,y)$ is arbitrary, we have

$$\frac{\partial f}{\partial z} - \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial z_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z_y}\right) = 0,$$

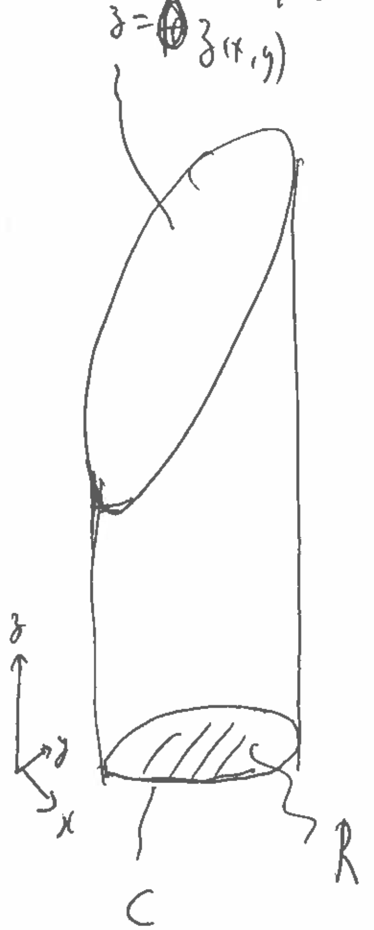
that is,
$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial z_x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z_y}\right) - \frac{\partial f}{\partial z} = 0. \quad (4.4)$$

This is the Euler equation for an extremal ~~of~~ the integral (4.1) subject to the given boundary condition on C.

Example 4.1 (The problem of minimal surface).

Find the surface of smallest area bounded by a closed curve ~~described by $z = z(x,y)$~~ in space.

Assume this curve projects down to a closed curve surrounding a region R in the xy-plane, and the surfaced is described by $z = z(x,y)$, $(x,y) \in R$.



The surface area

$$A = \iint_R \sqrt{1+z_x^2+z_y^2}\; dx\,dy,$$

so we need to minimize this integral subject to the boundary condition on C.

Since $f(x,y,z,z_x,z_y) = \sqrt{1+z_x^2+z_y^2}$,

the Euler equation (4.4) reduces to

$$\frac{\partial}{\partial x}\left(\frac{z_x}{\sqrt{1+z_x^2+z_y^2}}\right) + \frac{\partial}{\partial y}\left(\frac{z_y}{\sqrt{1+z_x^2+z_y^2}}\right) = 0$$

$$\Longrightarrow \frac{1}{1+z_x^2+z_y^2}\left[\sqrt{1+z_x^2+z_y^2}\cdot z_{xx} - z_x\cdot\left(\frac{2z_x\cdot z_{xx}+2z_y\cdot z_{yx}}{2\sqrt{1+z_x^2+z_y^2}}\right)\right] + \frac{1}{1+z_x^2+z_y^2}\cdot\left[\sqrt{1+z_x^2+z_y^2}\cdot z_{yy} - z_y\cdot\left(\frac{2\cdot z_x\cdot z_{xy}+2z_y\cdot z_{yy}}{2\sqrt{1+z_x^2+z_y^2}}\right)\right] = 0,$$

$$\overset{\text{simplifying}}{\Longrightarrow} \quad z_{xx}(1+z_y^2) - 2z_x z_y\cdot z_{xy} + z_{yy}(1+z_x^2) = 0. \qquad (4.5)$$

Complete Solution: T. Radó (1930) and J. Douglas (1931), independently.