3 Finite side conditions

Find a space curve $x = x(t)$, $y = y(t)$, $z = z(t)$ on the surface

$$G(x, y, z) = 0 \qquad (3.20)$$

that gives a stationary value to the integral

$$I = \int_{t_1}^{t_2} f(\dot{x}, \dot{y}, \dot{z})\, dt \qquad (3.21)$$

Assume $G_z \neq 0$, from (3.20) we solve $z = g(x, y)$, and

$$\dot{z} = \frac{\partial g}{\partial x} \cdot \dot{x} + \frac{\partial g}{\partial y} \cdot \dot{y} \qquad (3.22)$$

Then

$$I = \int_{t_1}^{t_2} f\left(\dot{x}, \dot{y}, \frac{\partial g}{\partial x} \cdot \dot{x} + \frac{\partial g}{\partial y} \cdot \dot{y}\right) dt .$$

Thus, the Euler equations are

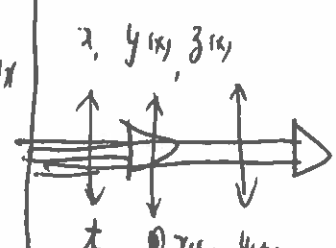
$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{\partial f}{\partial \dot{x}} + \dfrac{\partial f}{\partial \dot{z}} \dfrac{\partial g}{\partial x}\right) - \dfrac{\partial f}{\partial \dot{z}} \cdot \dfrac{\partial \dot{z}}{\partial x} = 0 \\[2ex] \dfrac{d}{dt}\left(\dfrac{\partial f}{\partial \dot{y}} + \dfrac{\partial f}{\partial \dot{z}} \cdot \dfrac{\partial g}{\partial y}\right) - \dfrac{\partial f}{\partial \dot{z}} \cdot \dfrac{\partial \dot{z}}{\partial y} = 0. \end{cases}$$

Recall that

$$I = \int_{x_1}^{x_2} f(x, y, z, y', z')\, dx$$

Euler equation:

$$\begin{cases} \dfrac{d}{dx}\left(\dfrac{\partial f}{\partial y'}\right) - \dfrac{\partial f}{\partial y} = 0 \\[2ex] \dfrac{d}{dx}\left(\dfrac{\partial f}{\partial z'}\right) - \dfrac{\partial f}{\partial z} = 0 \end{cases}$$



From (3.22) we see that

$$\frac{\partial \dot z}{\partial x} = \frac{\partial^2 g}{\partial x^2} \cdot \dot x + \frac{\partial^2 g}{\partial y \partial x} \cdot \dot y = \frac{d}{dt}\left(\frac{\partial g(x,y)}{\partial x}\right) = \frac{d}{dt}\left(\frac{\partial g}{\partial x}\right)$$

and
$$\frac{\partial \dot z}{\partial y} = \frac{\partial^2 g}{\partial x \partial y} \cdot \dot x + \frac{\partial^2 g}{\partial y^2} \cdot \dot y = \frac{d}{dt}\left(\frac{\partial g(x,y)}{\partial y}\right) = \frac{d}{dt}\left(\frac{\partial g}{\partial y}\right)$$

Euler equation

$$\Longrightarrow \quad \frac{d}{dt}\left(\frac{\partial f}{\partial \dot x}\right) + \frac{d}{dt}\left(\frac{\partial f}{\partial \dot z} \cdot \frac{\partial g}{\partial x}\right) - \frac{\partial f}{\partial \dot z} \cdot \frac{d}{dt}\left(\frac{\partial g}{\partial x}\right) = 0$$

$$\Longrightarrow \quad \frac{d}{dt}\left(\frac{\partial f}{\partial \dot x}\right) + \frac{d}{dt}\left(\frac{\partial f}{\partial \dot z}\right) \cdot \frac{\partial g}{\partial x} = 0 \qquad (3.23)$$

Similarly,
$$\frac{d}{dt}\left(\frac{\partial f}{\partial \dot y}\right) + \frac{d}{dt}\left(\frac{\partial f}{\partial \dot z}\right) \cdot \frac{\partial g}{\partial y} = 0 \qquad (3.24)$$

Define $\lambda(t)$ by
$$\frac{d}{dt}\left(\frac{\partial f}{\partial \dot z}\right) = \lambda(t) \cdot G_z . \qquad (3.25)$$

Note that $\frac{\partial g}{\partial x} = -\frac{G_x}{G_z}$ and $\frac{\partial g}{\partial y} = -\frac{G_y}{G_z}$ (why?).

Then (3.23) and (3.24) become
$$\frac{d}{dt}\left(\frac{\partial f}{\partial \dot x}\right) = \lambda(t)\, G_x , \qquad (3.26)$$
and
$$\frac{d}{dt}\left(\frac{\partial f}{\partial \dot y}\right) = \lambda(t)\, G_y . \qquad (3.27)$$

Thus, a necessary condition for a stationary value is the existence of a function $\lambda(t)$ satisfying (3.25), (3.26) and (3.27). On eliminating $\lambda(t)$, we have (symmetric form)

$$\frac{\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}}\right)}{G_x} = \frac{\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{y}}\right)}{G_y} = \frac{\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{z}}\right)}{G_z}. \quad (3.28)$$

This equation, together with (3.20), determine the extremes of the problem.

We remark that (3.25), (3.26) and (3.27) may be regarded as the Euler equations for the problem

$$\int_{t_1}^{t_2} [f(\dot{x}, \dot{y}, \dot{z}) + \lambda(t) G(x, y, z)]\, dt.$$

t, $\underline{x(t), y(t), z(t)}$, three functions of t.

x, $\underline{y(x), z(x)}$, two functions of x

Recall that

$$I = \int_{x_1}^{x_2} f(x, y, z, y', z')\, dx$$

$$J = \int_{x_1}^{x_2} g(x, y, z, y', z')\, dx = c$$

$$F = f + \lambda g$$

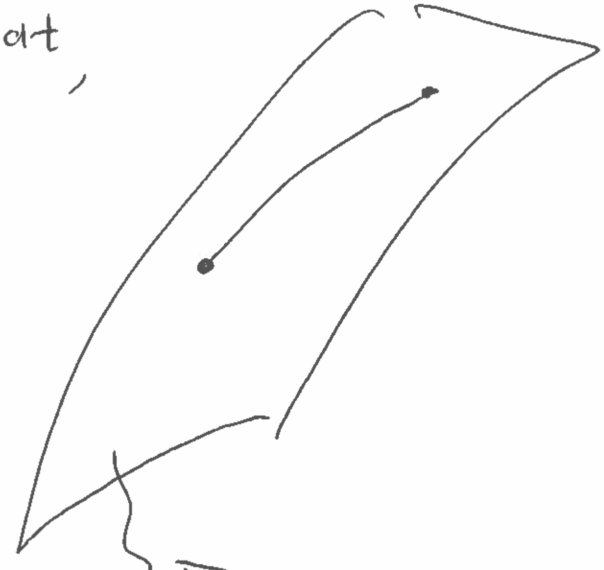
$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$$

$$\frac{d}{dx}\left(\frac{\partial F}{\partial z'}\right) - \frac{\partial F}{\partial z} = 0$$

For the <u>geodesics</u> on the surface $G(x,y,z)=0$,

$$I=\int_{t_1}^{t_2}\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}\,dt,$$

~~and hence, (3,28) becomes the the~~



we have $f=\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}$,

and hence (3,28) becomes

$$\frac{\frac{d}{dt}(\dot{x}/f)}{G_x}=\frac{\frac{d}{dt}(\dot{y}/f)}{G_y}=\frac{\frac{d}{dt}(\dot{z}/f)}{G_z}\qquad \Sigma:\ G(x,y,z)=0 \qquad (3.29)$$

<u>Example 3</u> For Σ: $x^2+y^2+z^2=a^2$, find geodesics.

<u>Solution.</u> $G(x,y,z)=x^2+y^2+z^2-a^2$. Then (3,29) is

$$\frac{f\cdot\ddot{x}-\dot{x}\dot{f}}{\cancel{2}x\cancel{f^2}}=\frac{f\cdot\ddot{y}-\dot{y}\dot{f}}{\cancel{2}y\cdot\cancel{f^2}}=\frac{f\ddot{z}-\dot{z}\dot{f}}{\cancel{2}z\cdot\cancel{f^2}}$$

$$(f\ddot{x}-\dot{x}\dot{f})y=x\cdot(f\ddot{y}-\dot{y}\dot{f})\implies f(x\ddot{y}-y\cdot\ddot{x})=\dot{f}(x\cdot\dot{y}-y\cdot\dot{x})$$

$$\implies \frac{x\ddot{y}-y\cdot\ddot{x}}{x\dot{y}-y\cdot\dot{x}}=\frac{f'}{f}\ \underset{\substack{\text{similarly,}\\ x\leftrightarrow y\\ y\leftrightarrow z}}{=\!=}\ \frac{y\cdot\ddot{z}-z\cdot\ddot{y}}{y\cdot\dot{z}-z\cdot\dot{y}}.$$

$$\Rightarrow \quad \frac{\frac{d}{dt}(x\dot{y} - y\cdot\dot{x})}{x\dot{y} - y\cdot\dot{x}} = \frac{\frac{d}{dt}(y\dot{z} - z\cdot\dot{y})}{y\cdot\dot{z} - z\cdot\dot{y}}$$

$$\Rightarrow \quad \frac{d}{dt}\ln(x\dot{y} - y\cdot\dot{x}) = \frac{d}{dt}\ln(y\dot{z} - z\cdot\dot{y})$$

$$\Rightarrow \quad x\dot{y} - y\dot{x} = c_1(y\dot{z} - z\cdot\dot{y})$$

$$\Rightarrow \quad \frac{\dot{x} + c_1\dot{z}}{x + c_1 z} = \frac{\dot{y}}{y}.$$

$$\Rightarrow \quad \frac{d}{dt}\ln(x + c_1 z) = \frac{d}{dt}\ln y$$

$$\Rightarrow \quad x + c_1 z = c_2\cdot y,$$

which is the equation of a <u>plane</u> through the <u>origin</u>.

Thus, the <u>geodesics</u> on a <u>sphere</u> are arcs of <u>great circles</u>.