$$= a_0 \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} + a_1 \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}. \qquad (1.6)$$

Here a_0 and a_1 are two constants

Note that $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$, $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$,

for all $x \in (-\infty, \infty)$.

Example 2 Find a series solution of Airy's equation

$$y'' - xy = 0, \quad -\infty < x < \infty \qquad (1.7)$$

Solution. For (1.7), we have $P(x) = 1$, $Q(x) = 0$, $R(x) = -x$,

and hence, every point is an ordinary point.

Let $y = \sum_{n=0}^{\infty} a_n x^n$, $|x| < \rho$. Then

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \xlongequal[n-2 \leftrightarrow n]{} \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} \cdot x^n.$$

$$\xRightarrow{(1.7)} \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n = x \cdot \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=0}^{\infty} a_n \cdot x^{n+1}$$

$$\overset{n+1 \leftrightarrow n}{=} \sum_{n=1}^{\infty} a_{n-1} x^n$$

$$\Longrightarrow \quad 2\cdot 1\cdot a_2 + \sum_{n=1}^{\infty} (n+2)(n+1)\cdot a_{n+2}\,x^n = \sum_{n=1}^{\infty} a_{n-1}\cdot x^n$$

$$\Longrightarrow \quad a_2 = 0, \quad \text{and} \quad (n+2)(n+1)\,a_{n+2} = a_{n-1}, \quad n = 1, 2, \cdots$$

i.e., $a_{n+2} = \dfrac{1}{(n+2)(n+1)}\cdot a_{n-1}$, $n = 1, 2, \cdots$

$= \dfrac{1}{(n+1)(n+2)}\cdot a_{n-1}$

Since $\overline{(n+2)-(n-1) = 3}$

$a_2 = 0$, $\Longrightarrow$ $a_5 = 0$, $a_8 = 0$, $a_{11} = 0$, $\cdots$

$\Longrightarrow$ $a_{3k+2} = 0$, $k \geq 0$

For a_0,

$a_3 = \dfrac{1}{2\cdot 3}\cdot a_0$, $\quad a_6 = \dfrac{1}{5\cdot 6}\cdot a_3 = \dfrac{1}{2\cdot 3\cdot 5\cdot 6}\, a_0$,

$a_9 = \dfrac{1}{8\cdot 9}\, a_6 = \dfrac{1}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9}\, a_0$, $\cdots$

$$\Longrightarrow \quad a_{3k} = \frac{a_0}{2\cdot 3\cdot 5\cdot 6\cdots (3k-1)\cdot (3k)}, \quad k \geq 1$$

For a_1,

$a_4 = \dfrac{a_1}{3\cdot 4}$, $\quad a_7 = \dfrac{a_4}{6\cdot 7} = \dfrac{a_1}{3\cdot 4\cdot 6\cdot 7}$,

$a_{10} = \dfrac{a_7}{9\cdot 10} = \dfrac{a_1}{3\cdot 4\cdot 6\cdot 7\cdot 9\cdot 10}$, $\cdots$

$$\Longrightarrow \quad a_{3k+1} = \frac{a_1}{3\cdot 4\cdot 6\cdot 7\cdot 9\cdot 10\cdots (3k)\cdot (3k+1)}, \quad k \geq 1.$$

Thus, $y = \sum_{k=0}^{\infty} a_{3k} \cdot x^{3k} + \sum_{k=0}^{\infty} a_{3k+1} \cdot x^{3k+1}$

$$= a_0\left[1+\sum_{k=1}^{\infty} \frac{1}{2\cdot 3\cdot 5\cdot 6 \cdots (3k-1)(3k)}\, x^{3k}\right]$$

$$+ a_1\left[x+\sum_{k=1}^{\infty} \frac{1}{3\cdot 4\cdot 6\cdot 7 \cdots (3k)\cdot(3k+1)}\, x^{3k+1}\right]$$

$$= a_0\left[1+\sum_{n=1}^{\infty} \frac{1}{2\cdot 3\cdot 5\cdot 6 \cdots (3n-1)\cdot(3n)} \cdot x^{3n}\right]$$

$$+ a_1\left[x+\sum_{n=1}^{\infty} \frac{1}{3\cdot 4\cdot 6\cdot 7 \cdots (3n)\cdot(3n+1)} \cdot x^{3n+1}\right] \quad (8)$$

$\Rightarrow$ $y = a_0 \cdot y_1(x) + a_1 \cdot y_2(x)$, $-\infty < x < \infty$

Note that $y_1(0) = 1$, $y_1'(0) = 0$

$y_2(0) = 0$, $y_2'(0) = 1$

Thus, $W[y_1, y_2](0) \overset{\text{def.}}{=} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1 \neq 0$, and hence,

$y_1(x)$ and $y_2(x)$ are <u>linearly independent.</u>