Recall
$$\underbrace{I(\alpha_1, \alpha_2)} = \int_{x_1}^{x_2} f(x, \bar{y}, \bar{y}')\,dx \qquad (3.11)$$

$$J(\alpha_1, \alpha_2) = \int_{x_1}^{x_2} g(x, \bar{y}, \bar{y}')\,dx = c \qquad (3.12)$$

$$\underbrace{J(\alpha_1, \alpha_2) - c} = 0$$

Let
$$K(\alpha_1, \alpha_2, \lambda) = I(\alpha_1, \alpha_2) + \lambda\left(J(\alpha_1, \alpha_2) - c\right)$$
$$= \int_{x_1}^{x_2} \left(f(x, \bar{y}, \bar{y}') + \lambda g(x, \bar{y}, \bar{y}')\right) dx - \lambda c$$
$$= \int_{x_1}^{x_2} F(x, \bar{y}, \bar{y}')\,dx - \lambda c,$$

where $\underline{F = f + \lambda g}$. When $(\alpha_1, \alpha_2) = (0, 0)$,

$$\Longrightarrow \quad \frac{\partial K}{\partial \alpha_1} = 0, \quad \frac{\partial K}{\partial \alpha_2} = 0, \quad \frac{\partial K}{\partial \lambda} = \underline{J(\alpha_1, \alpha_2) - c = 0} \qquad (3.13)$$

$$\Longrightarrow \frac{\partial K}{\partial \alpha_i} = \int_{x_1}^{x_2} \frac{\partial F}{\partial \bar{y}} \cdot \eta_i(x) + \frac{\partial F}{\partial \bar{y}'} \cdot \eta_i'(x) \Big]\,dx, \quad i = 1, 2.$$

Recall that $\bar{y}(x) = y(x) + \alpha_1 \eta_1(x) + \alpha_2 \eta_2(x)$, $\bar{y}'(x) = y'(x) + \alpha_1 \eta_1'(x) + \alpha_2 \eta_2'(x)$

Letting $(\alpha_1, \alpha_2) = (0, 0)$, we obtain $J(0,0) = \int_{x_1}^{x_2} g(x, y, y')\,dx = c$, and

$$\int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y} \cdot \eta_i(x) + \frac{\partial F}{\partial y'} \cdot \eta_i'(x) \right] dx = 0$$

Integrating by parts, we have

$$\int_{x_1}^{x_2} \eta_i(x) \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) \right] dx = 0 \qquad (3.14)$$

Since $\eta_1(x)$ and $\eta_2(x)$ are arbitrary, it follows that

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0. \qquad (3.15)$$

(Euler's equation)

Thus, we can selected the stationary function by solving (3.15) and $\int_{x_1}^{x_2} g(x, y, y')\,dx = c$ under the two boundary conditions.

Remark This idea can be extended to the case where the integrals depend on two or more functions. For example,

$$I = \int_{x_1}^{x_2} f(x, y, z, y', z')\,dx, \quad \text{and} \quad J = \int_{x_1}^{x_2} g(x, y, z, y', z')\,dx = c.$$

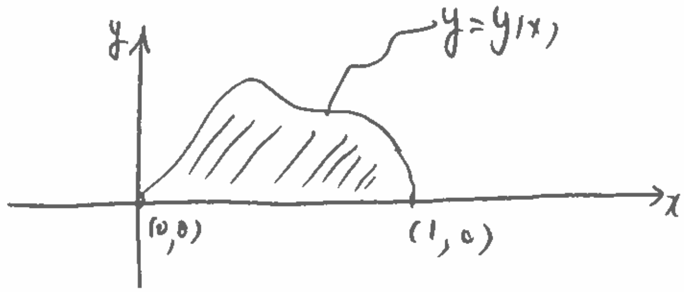
The stationary functions $y(x)$ and $z(x)$ must satisfy

$$\begin{cases} \dfrac{d}{dx}\left(\dfrac{\partial F}{\partial y'}\right) - \dfrac{\partial F}{\partial y} = 0 \\ \\ \dfrac{d}{dx}\left(\dfrac{\partial F}{\partial z'}\right) - \dfrac{\partial F}{\partial z} = 0, \end{cases} \qquad (3.16)$$

where $F = f + \lambda \cdot g$.

<u>Example 1</u> Find the curve of fixed length L that <u>joins</u> $(0,0)$ and $(1,0)$, <u>lies</u> above the x-axis, and <u>encloses</u> the <u>maximum area</u> between itself and the x-axis.

<u>Solution.</u>



We need to maximize $\int_0^1 y\,dx$ subject to the side condition $\int_0^1 \sqrt{1+(y')^2}\,dx = L$ and the boundary conditions $y(0)=0$ and $y(1)=0$. Thus, $F = f + \lambda \cdot g = y + \lambda\sqrt{1+(y')^2}$.

and the Euler equation (3.15) is

$$\frac{d}{dx}\left(\frac{\lambda \cdot y'}{\sqrt{1+(y')^2}}\right) - 1 = 0 \qquad (3.17)$$

$$\Rightarrow \frac{d}{dx}\left[\left(1+(y')^2\right)^{-\frac{1}{2}} \cdot y'\right] = \frac{1}{\lambda}$$

$$\Rightarrow \left(-\frac{1}{2}\right)\cdot\left(1+(y')^2\right)^{-\frac{3}{2}} \cdot 2y'\cdot y'' \cdot y' + \left(1+(y')^2\right)^{-\frac{1}{2}} \cdot y'' = \frac{1}{\lambda}$$

$$\Rightarrow \frac{y''}{\left[1+(y')^2\right]^{3/2}} = \frac{1}{\lambda} \qquad (3.18)$$

This implies that the curvature of $y(x)$ is $\frac{1}{\lambda}$, and hence, the required maximizing curve is an arc of a circle with radius λ.

Alternatively, we integrate (3.17) to get

$$\frac{y'}{\sqrt{1+(y')^2}} = (x-c_1)/\lambda$$

$$\Rightarrow \frac{(y')^2}{1+(y')^2} = \frac{(x-c_1)^2}{\lambda^2} \Rightarrow (y')^2 = \frac{(x-c_1)^2}{\lambda^2-(x-c_1)^2}$$

$$\Rightarrow y' = \pm\frac{x-c_1}{\sqrt{\lambda^2-(x-c_1)^2}} \Rightarrow y = \mp\left(\lambda^2-(x-c_1)^2\right)^{\frac{1}{2}} + c_2$$

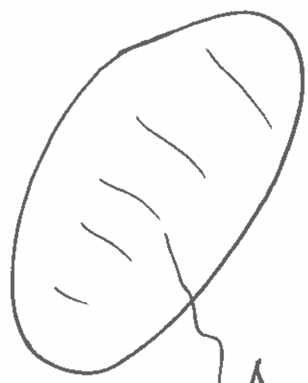
$$\Rightarrow (y-c_2)^2 = \lambda^2-(x-c_1)^2 \Rightarrow (x-c_1)^2+(y-c_2)^2 = \lambda^2, \qquad (3.19)$$

which is the equation of a circle with radius λ.

Example 2 (Isoperimetric problem)

Let $x = x(t)$, $t_1 \le t \le t_2$

$y = y(t)$, $t_1 \le t \le t_2$, we maximize $\left(\dot{x} = \frac{dx}{dt},\ \dot{y} = \frac{dy}{dt} \right)$



A

$$A = \frac{1}{2} \int_{t_1}^{t_2} (x \cdot \dot{y} - y \dot{x})\, dt$$

with the side condition $\int_{t_1}^{t_2} \sqrt{\dot{x}^2 + \dot{y}^2}\, dt = L$.

Then $F = \frac{1}{2}(x \cdot \dot{y} - y \dot{x}) + \lambda \sqrt{\dot{x}^2 + \dot{y}^2}$

By (3.16), we have

$x \leftrightarrow t$
$y \leftrightarrow x$
$z \leftrightarrow y$

$$\frac{d}{dt}\left(-\frac{1}{2} y + \frac{\lambda \dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}} \right) - \frac{1}{2}\dot{y} = 0$$

and

$$\frac{d}{dt}\left(\frac{1}{2} x + \frac{\lambda \dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}} \right) + \frac{1}{2}\dot{x} = 0$$

$\Longrightarrow$ $-y + \frac{\lambda \cdot \dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}} = -c_1$ and $x + \frac{\lambda \dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}} = c_2$

$\Longrightarrow$ $(x - c_2)^2 + (y - c_1)^2 = \lambda^2$. Thus, the maximizing curve is a <u>circle</u>

To determine λ, we let $2\pi \cdot \lambda = L$, $\Longrightarrow$ $\lambda = \frac{L}{2\pi}$, and hence

$A = \pi \cdot \lambda^2 = \frac{L^2}{4\pi}$.