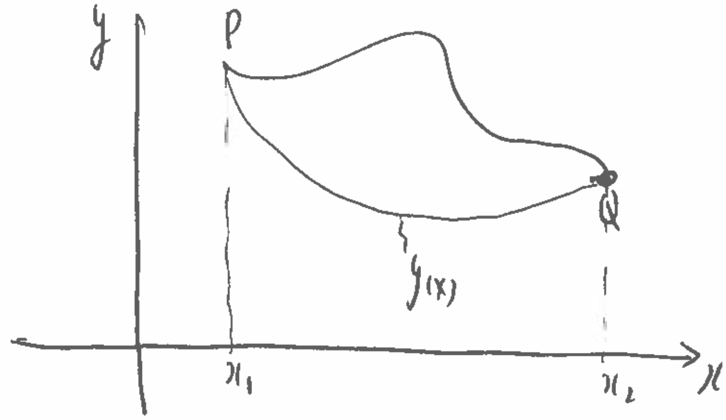
Chapt. 3 The Calculus of Variations

§1 Introduction

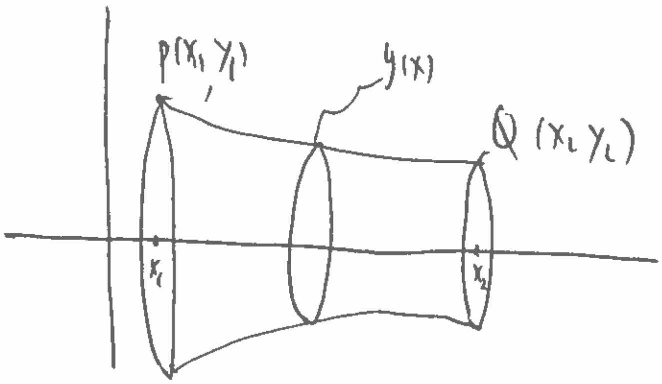
<u>Example 1</u> Let P and Q be two given points in a plane. Among curves joining P and Q, which one is the shortest?



Let $P = (x_1, y_1)$, $Q = (x_2, y_2)$. The curve is the graph of $y = y(x)$. Then the length is

$$L[y] = \int_{x_1}^{x_2} \sqrt{1+(y')^2}\, dx \quad , \quad \text{minimize } L[y]\ ?$$

<u>Example 2</u>. For all curves joining P and Q, which curve generates the surface of <u>revolution</u> of <u>smallest</u> area when revolved about the x-axis?

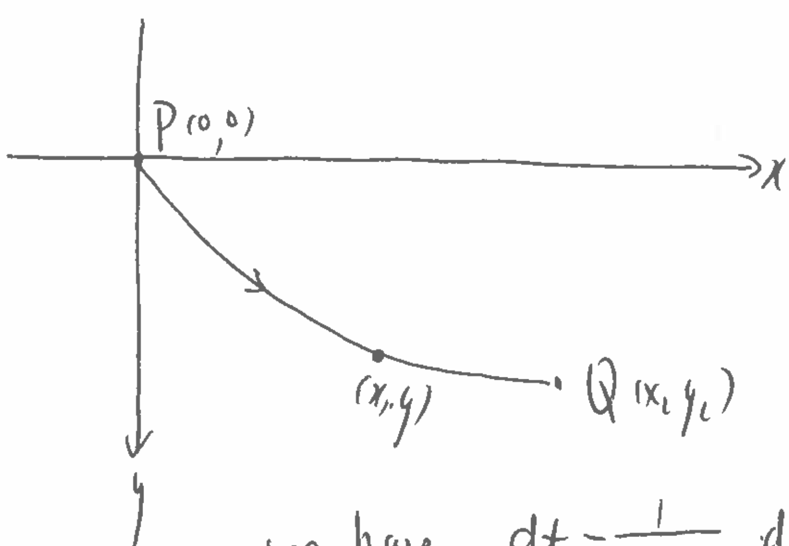


Let $y=y(x)$ be the curve joining P and Q. The the area of the surface of revolution is

$$A[y] = \int_{x_1}^{x_2} 2\pi \cdot y \cdot \sqrt{1+(y')^2}\, dx,$$

minimize $A[y]$?

<u>Example 3</u> Let a curve joining P and Q be a frictionless wire in a vertical plane, can we find the curve down which a bead will slide from P to Q in the shortest time ?



Let $y=y(x)$ be a curve joining P and Q. The speed $v = \sqrt{2g \cdot y}$

Since $\frac{ds}{dt} = v = \sqrt{2g \cdot y}$,

we have $dt = \frac{1}{\sqrt{2g \cdot y}}\, ds = \frac{1}{\sqrt{2g \cdot y}} \cdot \sqrt{1+(y')^2}\, dx$.

Thus the total time of descent from P to Q is

$$T[y] = \int_{x_1}^{x_2} \frac{\sqrt{1+(y')^2}}{\sqrt{2g \cdot y}}\, dx \quad , \quad \text{minimize } T[y] \,?$$

Note that these quantities depends on the curve (that is the function $y = y(x)$). The calculus of variation provides a uniform analytic method to deal with such minimum or maximum problems.

§2. Euler's Differential Equation for an Extremal

Consider

$$I = \int_{x_1}^{x_2} f(x, y, y')\, dx \qquad (2.1)$$

Assume an admissible function $y(x)$ minimizes I.

Let $\eta(x)$ be any function with continuous $\eta''(x)$ such that

$$\eta(x_1) = \eta(x_2) = 0 \qquad (2.2)$$

Consider
$$\bar{y}(x) = y(x) + \alpha\,\eta(x), \qquad (2.3)$$
$$\alpha \in (-\infty, \infty).$$

If α is small, $\bar{y}(x)$ is a <u>perturbation</u> of $y(x)$.
($\bar{y} - y = \alpha\eta$ is called the <u>variation</u> of the function y)

Then
$$I(\alpha) = \int_{x_1}^{x_2} f(x, \bar{y}, \bar{y}')\,dx$$
$$= \int_{x_1}^{x_2} f(x,\ y(x)+\alpha\cdot\eta(x),\ y'(x)+\alpha\eta'(x))\,dx \qquad (2.4)$$

when ~~Since~~ $\alpha = 0$, $\bar{y}(x) = y(x)$. Then $I(\alpha)$ must have a minimum when $\alpha = 0$. Thus, $I'(0) = 0$.

Since
$$I'(\alpha) = \int_{x_1}^{x_2} \frac{\partial}{\partial\alpha} f(x, \bar{y}, \bar{y}')\,dx$$
$$= \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial \bar{y}} \cdot \overset{\frac{\partial \bar{y}}{\partial\alpha}}{\eta(x)} + \frac{\partial f}{\partial \bar{y}'} \cdot \overset{\frac{\partial \bar{y}'}{\partial\alpha}}{\eta'(x)} \right] dx \qquad (2.5)$$

$$\Rightarrow\ I'(0) = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y}\cdot\eta(x) + \frac{\partial f}{\partial y'}\cdot\eta'(x) \right] dx = 0 \qquad (2.6)$$

Note that $\displaystyle\int_{x_1}^{x_2} \frac{\partial f}{\partial y'}\cdot\eta'(x)\,dx = \frac{\partial f}{\partial y'}\cdot\eta(x)\Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\cdot\eta(x)\,dx.$

$$\xRightarrow[(2.2)]{(2.6)} \quad \int_{x_1}^{x_2} \eta(x)\cdot\left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right] dx = 0$$

Since $\eta(x)$ is arbitrary, it follows from the argument by contradiction that

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} = 0, \qquad (2.7)$$

which is Euler's equation

Motivated by the elementary calculus we call any admissible solution of Euler equation a stationary function (or stationary curve), the corresponding value of the integral I as a stationary value of I. Further, solutions of Euler's equation unrestricted by the boundary conditions are called extremals.