§4. Singular Sturm-Liouville B.V.P.

Recall (S.-L. problem)
$$\begin{cases} L[y] = -[p(x)y']' + q(x)y = \lambda\, r(x)\cdot y, & 0<x<1 \\ \alpha_1 y(0) + \alpha_2 y'(0) = 0 \\ \beta_1 y(1) + \beta_2 \cdot y'(1) = 0 \end{cases}$$

where p, q, r are continuous, and $p(x)>0$, $r(x)>0$ on $[0,1]$.

Example 4.1 Consider
$$xy'' + y' + \lambda x y = 0 \quad (4.1)$$
$0<x<1$,

and suppose $\lambda>0$. $y(0)=0$ $y(1)=0$ (4.2)

$$(4.1) \Longrightarrow \quad -(xy')' = \lambda x y \quad (4.3)$$

$\Rightarrow p(x) = x$, $q(x)=0$ $r(x) = x$ ($p(0)=0$, $r(0)=0$)

Let $t = \sqrt{\lambda}\, x$, $x = \frac{1}{\sqrt{\lambda}} t$

$$\Longrightarrow \frac{dy}{dx} = \sqrt{\lambda}\cdot\frac{dy}{dt}, \quad \frac{d^2y}{dx^2} = \lambda\cdot\frac{d^2y}{dt^2}$$

$$\overset{(4.1)}{\Longrightarrow} \frac{t}{\sqrt{\lambda}}\cdot\lambda\cdot\frac{d^2y}{dt^2} + \sqrt{\lambda}\cdot\frac{dy}{dt} + \lambda\cdot\frac{t}{\sqrt{\lambda}}\cdot y = 0$$

$$\Longrightarrow \quad t\cdot\frac{d^2y}{dt^2} + \frac{dy}{dt} + t\cdot y = 0 \qquad (4.4)$$

$$\Longrightarrow \quad t^2\cdot\frac{d^2y}{dt^2} + t\cdot\frac{dy}{dt} + t^2\cdot y = 0$$ (Bessel's equation of order zero)

$$\Longrightarrow \quad y = c_1 J_0(t) + c_2\cdot Y_0(t), \quad t>0$$

$$\Longrightarrow \quad y(x) = c_1 J_0(\sqrt{\lambda}\,x) + c_2\cdot Y_0(\sqrt{\lambda}\,x), \quad x>0 \qquad (4.5)$$

where $$J_0(\sqrt{\lambda}\,x) = 1 + \sum_{m=1}^{\infty}\frac{(-1)^m\cdot(\sqrt{\lambda}\,x)^{2m}}{2^{2m}\cdot(m!)^2}, \quad x>0$$

and $$Y_0(\sqrt{\lambda}\,x) = \frac{2}{\pi}\cdot\left[\left(\gamma + \ln\frac{\sqrt{\lambda}\,x}{2}\right)J_0(\sqrt{\lambda}\,x) + \sum_{m=1}^{\infty}\frac{(-1)^{m+1}\cdot H_m\cdot(\sqrt{\lambda}\,x)^{2m}}{2^{2m}\cdot(m!)^2}\right], \quad x>0.$$

Recall $H_m = 1+\frac{1}{2}+\cdots+\frac{1}{m}$ and $\gamma = \lim_{m\to\infty}(H_m - \ln m)$.

Note that $J_0(0)=1$ and $\lim_{x\to 0} Y_0(x) = -\infty$. Thus, the condition $y(0)=0$ can be satisfied only by choosing $c_1 = c_2 = 0$.

Thus the B.V.P. (4.1) and (4.2) has only the trivial solution.

Now we revise the ~~boun~~ boundary condition (4.2) as

$$\begin{cases} y \text{ and } y' \text{ are bounded as } x \to 0 \\ y(1) = 0 \end{cases} \qquad (4.6)$$

$\Longrightarrow$ $c_2 = 0$, and $y(1) = 0$ leads to $(c_1 \neq 0)$

$$J_0(\sqrt{\lambda}) = 0 \qquad (4.7)$$

$\Longrightarrow$ (4.7) has positive solutions $0 < \lambda_1 < \lambda_2 < \cdots\cdots$, and

$$\varphi_n(x) = J_0(\sqrt{\lambda_n}\, x), \quad \forall n \geq 1. \qquad (4.8)$$

Thus, the problem (4.1) and (4.6) has eigenvalue λ_n and eigenfunction $\varphi_n(x)$, $\forall n \geq 1$.

Definition 4.1 For $\underbrace{L[y] = -(p(x)y')' + q(x)y = \lambda r(x) y}_{\text{the problem}}$, if p, q, r are continuous on $(0,1)$, ~~and~~ and $p(x)>0$ $r(x)>0$ on $(0,1)$, but at least one of these functions fails to satisfies them at one or both of the boundary points, then it is called singular Sturm–Liouville problem.

Motivated by the regular S.-L. problem, we consider

$$\int_0^1 \Big(L[u]v - u \cdot L[v] \Big) dx = 0 \qquad (4.9)$$

Assume that the ~~p~~ S.-L. problem is singular at $x=0$

$$\Longrightarrow \underset{(0<\varepsilon<1)}{} \int_\varepsilon^1 \Big(L[u]\cdot v - u\cdot L[v] \Big) dx \xlongequal[\text{by parts}]{\text{integrating twice}} - p(x)\left[u'(x)v(x) - u(x)v'(x) \right]\Big|_\varepsilon^1$$

$$\xlongequal{\text{B.C. at } x=1} p(\varepsilon)\cdot\left[u'(\varepsilon)v(\varepsilon) - u(\varepsilon)v'(\varepsilon) \right] \qquad (4.10)$$

Thus, we impose the following condition

$$\lim_{\varepsilon \to 0} p(\varepsilon) \cdot \left[u'(\varepsilon) \cdot v(\varepsilon) - u(\varepsilon) \cdot v'(\varepsilon) \right] = 0 \qquad (4,11)$$

It then follows that (using <u>improper</u> integral)

$$\int_0^1 \left(L[u] \cdot v - u \cdot L[v] \right) dx = \lim_{\varepsilon \to 0} \int_\varepsilon^1 \left(L[u] \cdot v - u \cdot L[v] \right) dx$$

$$= 0$$

provided (4,11) is satisfied.

A similar condition applies at $x = 1$ if that boundary point is singular, namely,

$$\lim_{\varepsilon \to 0} p(1-\varepsilon) \left[u'(1-\varepsilon) v(1-\varepsilon) - u(1-\varepsilon) \cdot v'(1-\varepsilon) \right] = 0 \qquad (4,12).$$

For example, $(4,6) \Longrightarrow (4,11)$ at $x = 0$

(since $p(x) = x$)

<u>Remark 4.1</u> If a singular S.-L. problem has only a <u>discrete</u> set of eigenvalues and eigenfunctions, then (4.9) can be used to prove that ~~the~~ eigenvalues of such a problem are real and that the eigenfunctions are <u>orthogonal</u> w.r.t. the weight function $r(x)$. The expansion of a given function then follows