§3 Generalized Fourier Series and Eigenfunction Expansions

Recall that if f and f' are piecewise continuous on $[0,1]$ with $f(0)=f(1)$, then f has an Fourier series expansion:

$$f(x) = \sum_{n\geq 1}^{\infty} b_n \underline{\sin(n\pi x)}, \quad b_n = 2\int_0^1 f(x)\cdot \sin(n\pi x)\,dx, \quad \forall n \geq 1$$

Note that $\sin(n\pi x)$, $n = 1, 2, \cdots$, are the <u>eigenvalues</u> of

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(1) = 0$$

Consider a more general Sturn-Liouville problem

$$(p(x)y')' - q(x)y + \lambda r(x) y = 0 \tag{3.1}$$

$$\alpha_1 y(0) + \alpha_2 y'(0) = 0, \quad \beta_1 y(1) + \beta_2 \cdot y'(1) = 0 \tag{3.2}$$

<u>Assume that</u>

$$f(x) = \sum_{n\geq 1}^{\infty} c_n \varphi_n(x), \tag{3.3}$$

where $\varphi_n(x)$ is the <u>normalized</u> eigenfunction associated with eigenvalue λ_n, $n \geq 1$. Then

$$\langle f, r\varphi_m \rangle = \langle \sum_{n\geq 1}^{\infty} c_n \varphi_n, r\varphi_m \rangle = \sum_{n\geq 1}^{\infty} c_n \langle \varphi_n, r\varphi_m \rangle$$

$$= \sum_{n\geq 1}^{\infty} c_n \, \delta_{mn} = c_m \qquad (3.4)$$

Thus, $$c_m = \langle f, r\varphi_m \rangle = \int_0^1 r(x) f(x) \varphi_m(x)\, dx, \quad \forall m \geq 1. \qquad (3.5)$$

Example 3.1 Expand the function $f(x) = x$, $0 \leq x \leq 1$, in terms of the normalized eigenfunction $\varphi_n(x)$ of the S.-L. problem

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(1) + y(1) = 0. \qquad (3.6)$$

Solution We have found that

$$\varphi_n(x) = k_n \cdot \sin(\sqrt{\lambda_n}\, x), \qquad (3.7)$$

$$k_n = \left(\frac{2}{1 + \cos^2(\sqrt{\lambda_n})} \right)^{\frac{1}{2}} \qquad (3.8)$$

$$\sin\sqrt{\lambda_n} + \sqrt{\lambda_n} \cos\sqrt{\lambda_n} = 0 \qquad (3.9)$$

Then $$f(x) = \sum_{n\geq 1}^{\infty} c_n \cdot \varphi_n(x) \qquad (3.10)$$

$$c_n = \langle f, r\varphi_n \rangle \underset{r(x)\equiv 1}{=\!=\!=} k_n \cdot \int_0^1 x \cdot \sin(\sqrt{\lambda_n}\, x)\, dx$$

$$= k_n \left[x \cdot \left(-\frac{1}{\sqrt{\lambda_n}} \cos(\sqrt{\lambda_n}\, x) \right) \Big|_0^1 + \int_0^1 \frac{1}{\sqrt{\lambda_n}} \cdot \cos(\sqrt{\lambda_n}\, x)\, dx \right]$$

$$= k_n \left[-\frac{1}{\sqrt{\lambda_n}} \cos(\sqrt{\lambda_n}) + \frac{1}{\lambda_n} \sin(\sqrt{\lambda_n}\, x) \Big|_0^1 \right] = k_n \cdot \left[\frac{1}{\lambda_n} \sin(\sqrt{\lambda_n}) - \frac{1}{\sqrt{\lambda_n}} \cos(\sqrt{\lambda_n}\right.$$

$$\overset{(3.9)}{=\!=} k_n \cdot \frac{2\sin(\sqrt{\lambda_n})}{\lambda_n} \overset{(3.8)}{=\!=} \frac{\sqrt{2}}{\left(1+\cos^2(\sqrt{\lambda_n})\right)^{\frac{1}{2}}} \cdot \frac{2\sin(\sqrt{\lambda_n})}{\lambda_n}$$

$$= \frac{2\sqrt{2}\,\sin(\sqrt{\lambda_n})}{\lambda_n \cdot \left(1+\cos^2(\sqrt{\lambda_n})\right)^{\frac{1}{2}}} \qquad (3.11)$$

Thus

$$f(x) = \sum_{n=1}^{\infty} C_n \cdot \varphi_n(x) = \sum_{n=1}^{\infty} \frac{4 \cdot \sin(\sqrt{\lambda_n})}{\lambda_n \cdot \left(1+\cos^2(\sqrt{\lambda_n})\right)} \cdot \sin(\sqrt{\lambda_n}\, x) \qquad (3.12)$$

Next, we consider the nonhomogeneous S.-L. problem:

$$L[y] = -(p(x)\,y')' + q(x)\,y = \mu \cdot r(x)\,y + f(x), \qquad (3.13)$$

where μ is a given constant and f is a given function on $[0,1]$, and the boundary conditions are (homogeneous)

$$\alpha_1 y(0) + \alpha_2 y'(0) = 0, \quad \beta_1 y(1) + \beta_2 y'(1) = 0 \qquad (3.14)$$

Assume that p, p', q and r are continuous on $[0,1]$ and $p(x) > 0$, $r(x) > 0$ on $[0,1]$

Let $\lambda_1 < \lambda_2 < \lambda_3 \dots < \lambda_n < \dots$ be the eigenvalues of

$$L[y] = \lambda\, r(x)\, y \qquad (3.14)$$

subject to the condition (3.14), and $\varphi_1, \varphi_2, \dots \varphi_n, \dots$ be the corresponding <u>normalized</u> eigenfunctions.

Let $$\varphi(x) = \sum_{n=1}^{\infty} b_n \varphi_n(x) \qquad (3.15)$$

be a solution of (3.13)–(3.14). Then

$$L[\varphi](x) = L\left[\sum_{n=1}^{\infty} b_n \varphi_n\right](x) = \sum_{n=1}^{\infty} b_n \cdot \lambda_n \cdot r(x) \varphi_n(x) \qquad (3.16)$$

$$\overset{(3.13)}{\Longrightarrow} \quad \sum_{n=1}^{\infty} b_n \cdot \lambda_n \cdot r(x) \varphi_n(x) = \mu \cdot r(x) \sum_{n=1}^{\infty} b_n \varphi_n(x) + f(x)$$

Assume $$\frac{f(x)}{r(x)} = \sum_{n=1}^{\infty} c_n \cdot \varphi_n(x) \qquad (3.17)$$

then $$c_n = \left\langle \frac{f(x)}{r(x)}, r\varphi_n \right\rangle = \int_0^1 f(x) \varphi_n(x)\, dx, \quad \forall n \geq 1 \qquad (3.18)$$

Thus, $$\sum_{n\geq 1}^{\infty} b_n \lambda_n \cdot r(x) \cdot \varphi_n(x) = \mu \cdot r(x) \cdot \sum_{n\geq 1}^{\infty} b_n \varphi_n(x) + r(x) \cdot \sum_{n\geq 1}^{\infty} c_n \varphi_n(x)$$

$$\Longrightarrow \sum_{n=1}^{\infty} [(\lambda_n - \mu) b_n - c_n] \varphi_n(x) = 0$$

$$\Longrightarrow \quad (\lambda_n - \mu) b_n - c_n = 0, \quad \forall n \geq 1 \qquad (3.19)$$

In the case where $\mu \neq \lambda_n$, $\forall n \geq 1$, then

$$b_n = \frac{c_n}{\lambda_n - \mu}, \quad n \geq 1. \qquad (3.20)$$

and $$y = \varphi(x) = \sum_{n=1}^{\infty} \frac{c_n}{\lambda_n - \mu} \cdot \varphi_n(x). \qquad (3.21)$$