Theorem 2.2. Let φ_i be the eigenfunction of (2.1)–(2.2) with eigenvalue λ_i $(i=1,2)$. If $\lambda_1 \neq \lambda_2$, then φ_1 and φ_2 are orthogonal w.r.t. the weight function $r(x)$ in the sense that

$$\int_0^1 r(x)\varphi_1(x)\cdot\varphi_2(x)\,dx = 0 \qquad (2.6)$$

Proof. Since $L[\varphi_1] = \lambda_1 r\varphi_1$ and $L[\varphi_2] = \lambda_2 r\varphi_2$,

$$\underset{\substack{u=\varphi_1\\ v=\varphi_2}}{\overset{(2.5)}{\Longrightarrow}} \quad \langle \lambda_1 r\varphi_1, \varphi_2\rangle - \langle \varphi_1, \lambda_2 r\varphi_2\rangle = 0.$$

Note that λ_i, φ_i and r are all real. We then have

$$\Rightarrow \quad \lambda_1\int_0^1 r(x)\varphi_1(x)\cdot\varphi_2(x)\,dx - \lambda_2\cdot\int_0^1 \varphi_1(x)\cdot r(x)\cdot\varphi_2(x)\,dx = 0$$

$$\Rightarrow \quad (\lambda_1-\lambda_2)\int_0^1 r(x)\varphi_1(x)\,\varphi_2(x)\,dx = 0 \qquad (2.7)$$

$$\overset{\lambda_1\neq\lambda_2}{\Longrightarrow} \quad \int_0^1 r(x)\varphi_1(x)\varphi_2(x)\,dx = 0$$

Define

$$\langle u, v\rangle_r = \int_0^1 r(x)\,u(x)\,v(x)\,dx$$

$\|u\|_r^2 = \langle u, u\rangle_r$, and hence $\|u\|_r = \sqrt{\langle u,u\rangle_r}$.

Thus, (2.6) implies $<\varphi_1, \varphi_2>_r = 0$.

Theorem 2.3 The eigenvalues of S.-L. problem (2.1)—(2.2) are all simple, and the eigenvalues can be arranged as

$$\lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_n < \cdots$$

such that $\lim_{n\to\infty} \lambda_n = \infty$.

Let φ_n be the eigenfunction associated with λ_n.

Define $\psi_n = \dfrac{\varphi_n}{\|\varphi_n\|_r}$ (normalized eigenfunction)

Then

$$\|\psi_n\|_r^2 = <\psi_n, \psi_n>_r = \int_0^1 r(x)\cdot\psi_n^2(x)\,dx$$

$$= \frac{1}{\|\varphi_n\|_r^2}\cdot\int_0^1 r(x)\cdot\varphi_n^2(x)\,dx = \frac{1}{\|\varphi_n\|_r^2} <\varphi_n, \varphi_n>_r$$

$$= 1, \tag{2.8}$$

and hence $\|\psi_n\|_r = 1$.

Define the Kronecker delta as $\delta_{mn} = \begin{cases} 0, & \text{if } m \neq n \\ 1, & \text{if } m = n. \end{cases}$

$$\xRightarrow{(2.6)+(2.8)} \quad <\psi_m, \psi_n>_r = \int_0^1 r(x)\psi_m(x)\cdot\psi_n(x)\,dx = \delta_{mn}. \tag{2.9}$$

Example 2.1 Find the eigenvalues and the corresponding normalized eigenfunctions of $y'' + \lambda y = 0$, $y(0) = 0$, $y'(1) + y(1) = 0$.

Solution By Theorem 2.1, all eigenvalues are real.

In the case where $\lambda = 0$, $y = c_1 x + c_2$, and two boundary conditions require that $c_2 = 0$ and $c_1 + c_1 + c_2 = 0 \Longrightarrow c_1 = c_2 = 0$

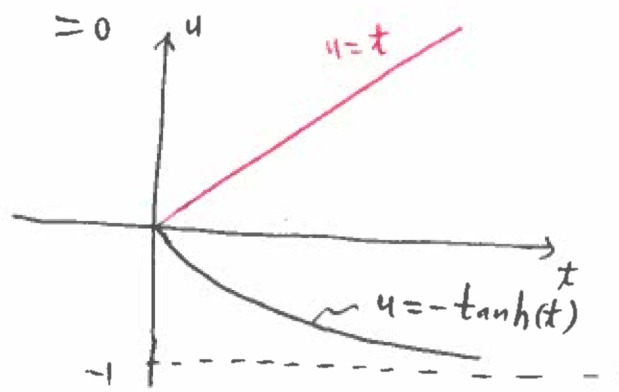
Thus, the problem has no nontrivial solution, and hence $\lambda = 0$ is not an eigenvalue.

In the case where $\lambda < 0$, let $\lambda = -\mu$, $\mu > 0$. Then

$$y'' - \mu y = 0 \Longrightarrow y = c_1 \sinh(\sqrt{\mu}\, x) + c_2 \cosh(\sqrt{\mu}\, x)$$

Then $y(0) = c_2 = 0$, and $y'(1) + y(1) = c_1 \cdot \sqrt{\mu} \cosh(\sqrt{\mu}) + c_1 \sinh(\sqrt{\mu}) = 0$

$$\xrightarrow{c_1 \neq 0} \sqrt{\mu} = -\tanh(\sqrt{\mu})$$



But this equation has no positive solution for $\sqrt{\mu}$. Thus, the problem has no negative eigenvalues.

By Theorem 2.3, the S.-L. problem has positive eigenvalues given by

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$$

In the case $\lambda > 0$, $\quad y = c_1 \sin(\sqrt{\lambda}\, x) + c_2 \cos(\sqrt{\lambda}\, x)$

Then $y(0) = c_2 = 0$, and $y'(1) + y(1) = c_1 \cdot \sqrt{\lambda} \cdot \cos(\sqrt{\lambda}) + c_1 \sin(\sqrt{\lambda}) = 0$

Thus, $\quad \sin(\sqrt{\lambda_n}) + \sqrt{\lambda_n} \cos(\sqrt{\lambda_n}) = 0 \quad$ (since $c_1 \neq 0$)

and $\quad \varphi_n(x) = \sin(\sqrt{\lambda_n}\, x) \quad$ (choose $c_1 = 1$)

For $y'' + \lambda y = 0$, $\quad p(x) = 1$, $q(x) = 0$, $r(x) = 1$.

Then $\|\varphi_n\|_1^2 = \langle \varphi_n, \varphi_n \rangle_1 = \int_0^1 \sin^2(\sqrt{\lambda_n}\, x)\, dx$

$\sin^2 t = \dfrac{1-\cos(2t)}{2}$

$$= \int_0^1 \frac{1 - \cos(2\sqrt{\lambda_n} \cdot x)}{2}\, dx$$

$$= \frac{1}{2}\left(x - \frac{1}{2\sqrt{\lambda_n}} \cdot \sin(2\sqrt{\lambda_n} \cdot x)\right)\Bigg|_0^1$$

$\sin(2t) = 2\sin t \cdot \cos t$

$$= \frac{1}{2}\left(1 - \frac{1}{2\sqrt{\lambda_n}} \sin(2\sqrt{\lambda_n})\right) = \frac{1}{2}\left(1 - \frac{\sin(\sqrt{\lambda_n}) \cdot \cos(\sqrt{\lambda_n})}{\sqrt{\lambda_n}}\right)$$

$$= \frac{\sqrt{\lambda_n} - \sin(\sqrt{\lambda_n}) \cdot \cos(\sqrt{\lambda_n})}{2\sqrt{\lambda_n}} = \frac{1 + \cos^2(\sqrt{\lambda_n})}{2}$$

Thus, $\psi_n(x) = \dfrac{1}{\|\varphi_n\|_1} \cdot \varphi_n = \dfrac{\sqrt{2}}{(1 + \cos^2(\sqrt{\lambda_n}))^{\frac{1}{2}}} \cdot \sin(\sqrt{\lambda_n} \cdot x)$, $\forall n \geq 1$.