Math 3161 (ODE II)

Chapt. 1 Series solutions to initial value problems

Reading ex. Review of power series (§9.1 in BB book)

§1. Series solutions near an ordinary point (I)

Consider $$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0 \qquad (1.1)$$

e.g., Airy equation $y'' - x\cdot y = 0$

Definition 1.1 A real number x_0 is said to be an ordinary point of (1.1) if P, Q and R are analytic and $P(x_0) \neq 0$. Otherwise, it is called a singular point of the equation.

If x_0 is an ordinary point, then we can divide (1.1) by $P(x)$ to obtain

$$y'' + p(x)y' + g(x)y = 0, \qquad (1.2)$$

where $p(x) = Q(x)/P(x)$ and $g(x) = \frac{R(x)}{P(x)}$ are analytic at x_0.

Example 1 Find a series solution of $y'' + y = 0$, $-\infty < x < \infty$. (1.3)

Solution

Since $P(x)=1$, $Q(x)=0$, $R(x)=1$, every x value is an ordinary point. We look for a solution of a power series about $x_0=0$:

$$y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots = \sum_{n\geq 0}^{\infty} a_n x^n.$$

Then

$$y' = a_1 + 2a_2 x + \cdots + n\cdot a_n x^{n-1} + \cdots = \sum_{n\geq 1}^{\infty} n\, a_n x^{n-1}$$

$$y'' = 2a_2 + \cdots n(n-1)\cdot a_n x^{n-2} + \cdots = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

$$\Rightarrow \quad \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\xrightarrow[\substack{n=m+2\\ n-1=m+1}]{\text{Let } n-2=m} \quad \sum_{m=0}^{\infty} (m+2)\cdot(m+1) a_{m+2} x^m + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \quad \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n\geq 0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \quad \sum_{n\geq 0}^{\infty} \left[(n+2)(n+1) a_{n+2} + a_n\right] x^n = 0$$

$$\Rightarrow \quad (n+2)(n+1) a_{n+2} + a_n = 0, \qquad n = 0, 1, 2, \cdots$$

$$\Rightarrow \quad a_{n+2} = -\frac{a_n}{(n+2)(n+1)}, \quad \forall\, n \geq 0.$$

"$\forall$" for all

For the even-numbered coefficients,

$$\Rightarrow a_2 = -\frac{a_0}{2\cdot 1} = -\frac{a_0}{2!}, \quad a_4 = -\frac{a_2}{4\cdot 3} = +\frac{a_0}{4!},$$

$$a_6 = -\frac{a_4}{6\cdot 5} = -\frac{a_0}{6!}, \quad \cdots$$

<u>By induction</u>, if $n = 2k$, then

$$a_{2k} = \frac{(-1)^k}{(2k)!} \cdot a_0, \quad k = 1, 2, \cdots \tag{1.4}$$

Similarly, for the odd-numbered coefficients,

$$a_3 = -\frac{a_1}{3\cdot 2} = -\frac{a_1}{3!}, \quad a_5 = -\frac{a_3}{5\cdot 4} = +\frac{a_1}{5!},$$

$$a_7 = -\frac{a_5}{7\cdot 6} = -\frac{a_1}{7!}, \cdots$$

<u>By induction</u>, if $n = 2k+1$, then

$$a_{2k+1} = \frac{(-1)^k}{(2k+1)!}\, a_1, \quad k = 1, 2, \cdots \tag{1.5}$$

Thus,

$$y = \sum_{n=0}^{\infty} a_n x^n = \sum_{k=0}^{\infty} a_{2k} x^{2k} + \sum_{k=0}^{\infty} a_{2k+1} \cdot x^{2k+1}$$

$$= a_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} + a_1 \cdot \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

$$= a_0 \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} + a_1 \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}. \tag{1.6}$$

Here a_0 and a_1 are two constants

Note that $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$, $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$,

for all $x \in (-\infty, \infty)$.

Example 2 Find a series solution of Airy's equation

$$y'' - xy = 0, \quad -\infty < x < \infty \tag{1.7}$$

Solution For (1.7), we have $P(x) = 1$, $Q(x) = 0$, $R(x) = -x$, and hence, every point is an ordinary point.

Let $y = \sum_{n=0}^{\infty} a_n x^n$, $|x| < \rho$. Then

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \overset{n-2 \leftrightarrow n}{=\!=\!=} \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} \cdot x^n.$$

$$\overset{(1.7)}{\Longrightarrow} \sum_{n=0}^{\infty} (n+2)(n+1) \cdot a_{n+2} x^n = x \cdot \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=0}^{\infty} a_n \cdot x^{n+1}$$

$n+1 \leftrightarrow n$

$$= \sum^{\infty} a_{n-1} x^n$$