Asymptotic Speeds of Spread and Traveling Waves for Monotone Semiflows with Applications

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Abstract

The theory of asymptotic speeds of spread and monotone traveling waves is established for a class of monotone discrete and continuous-time semiflows and is applied to a functional differential equation with diffusion, a time-delayed lattice population model and a reaction-diffusion equation in an infinite cylinder.

1 Introduction

Since the pioneering work of Fisher [14] and Kolmogorov, Petrovskii, and Piskunov [18], there have been extensive investigations on traveling wave solutions and asymptotic (long time) behavior in terms of spreading speeds for various evolution systems. Traveling waves were studied for nonlinear reaction-diffusion equations modeling physical and biological phenomena (see, e.g., books [26, 27, 42] and references therein), for integral and integrodifferential population models (see, e.g., [4, 7, 11, 13, 35]), for lattice differential systems (see, e.g., [5, 8, 9, 10, 23, 49, 53]), and for time-delayed reaction-diffusion equations (see, e.g., [34, 37, 40, 50]).

The concept of asymptotic speeds of spread was introduced by Aronson and Weinberger [1, 2, 3] for reaction-diffusion equations and applied by Aronson [1] to an integrodifferential equation. It was extended to a larger class of integral equations by Diekmann [12] and Thieme [38, 39] independently. In [44, 45], Weinberger proved the existence of asymptotic speeds of spread for a discrete-time recursion with a translation-invariant order-preserving operator. Radcliffe and Rass [29, 30, 31] studied traveling waves and asymptotic speeds of spread for a class of epidemic systems of integral equations (see also their book [32]). In [21, 22], Lui also generalized the results in [45] to systems of recursions.

Recently Weinberger, Lewis, and Li [19, 20, 47] extended the theory of spreading speeds and monotone traveling waves in [21, 45] in such a way that they can be applied to invasion processes of certain models for cooperation or competition among multiple species, and Weinberger [46] has also developed the theory in [21,

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45] to the order-preserving operators with a periodic habitat. Moreover, Thieme and Zhao [40] have generalized the earlier theory in [1, 4, 7, 11, 12, 13, 38, 39] to a class of nonlinear integral equations that is large enough to cover many time-delayed reaction-diffusion population models.

However, the theory for discrete-time recursions cannot be applied to autonomous time-delayed reaction-diffusion equations and lattice systems. This is because the solution map Q_t associated with such an equation is defined on the set of bounded and continuous functions from $[-\tau, 0] \times \mathcal{H}$ to $\mathbb{R}^k$, where $\mathcal{H}$ is the spatial habitat and τ is the time delay, and Q_t is not compact for $t \in (0, \tau)$ with respect to the compact open topology.

We also note that the theory developed in [40] applies only to scalar time-delayed reaction-diffusion equations and to certain types of reaction-diffusion systems with or without time delays that can be reduced to the scalar integral equations (see [40, 43]). Moreover, both discrete-time recursions and continuous-time integral equations approaches cannot be employed to study the spreading speeds and traveling waves for parabolic equations in infinite cylinders. We should point out that the spreading speed c^* and the existence of traveling waves with wave speed $c > c^*$ were established in [48] for a nonlocal time-delayed lattice system, and traveling waves were studied in [6, 24, 33, 41] for some parabolic equations in cylinders.

The purpose of this paper is to establish the theory of asymptotic speeds of spread and monotone traveling waves for monotone discrete and continuous-time semiflows with monostable nonlinearities so that it applies to the aforementioned evolution systems with time delays and reaction-diffusion equations in cylinders. Our methods and arguments are highly motivated by the earlier works in [21, 45]. However, this generalization is nontrivial and needs some new ideas and techniques such as the equicontinuity of the iterated sequences of functions, linear operators defined on an extended function space, the discrete-time maps approach to continuous-time semiflows, and the monotonicity and continuity of wave profiles for continuous-time semiflows with discrete spatial habitats.

Note that in the statement of the general theorem on spreading speeds, it is often assumed that the initial data $u_0(x) \geq \sigma$ on a ball of radius r_σ. We prove that r_σ can be chosen to be independent of the positive real number σ in the case where the monotone map Q *either* is subhomogeneous *or* can be approximated from below by a sequence of linear operators. Under a weaker compactness assumption on monotone discrete and continuous-time semiflows, we establish the existence of minimal wave speeds for monotone traveling waves and show that they coincide with the asymptotic speeds of spread.

The organization of this paper is as follows: In Section 2 we show the existence of asymptotic speeds of spread for monotone discrete and continuous-time semiflows. In Section 3 we give the estimates of spreading speeds by the linear operators approach. Section 4 establishes the existence of traveling waves above the spreading speeds and their nonexistence below the spreading speeds. In Section 5 we apply the theory in Sections 2 through 4 to a functional differential equation with diffusion, a nonlocal and time-delayed lattice population model, and a reaction-diffusion equation in a cylinder.

2 Asymptotic Speeds of Spread

Let τ be a nonnegative real number and $\mathcal{C}$ be the set of all bounded and continuous functions from $[-\tau, 0] \times \mathcal{H}$ to $\mathbb{R}^k$, where $\mathcal{H} = \mathbb{R}$ or $\mathbb{Z}$. Clearly, any vector in $\mathbb{R}^k$ and any element in the space $\bar{\mathcal{C}} := C([-\tau, 0], \mathbb{R}^k)$ can be regarded as a function in $\mathcal{C}$.

For $u = (u^1, \dots, u^k)$ and $v = (v^1, \dots, v^k) \in \mathcal{C}$, we write $u \geq v$ ($u \gg v$) provided $u^i(\theta, x) \geq v^i(\theta, x)$ ($u^i(\theta, x) > v^i(\theta, x)$) $\forall i = 1, \dots, k, \theta \in [-\tau, 0]$, and $x \in \mathcal{H}$; and $u > v$ provided $u \geq v$ but $u \neq v$. For any two vectors a and b in $\mathbb{R}^k$ or two functions $a, b \in \bar{\mathcal{C}}$, we can define $a \geq (>, \gg)\, b$ similarly. For any $r \in \bar{\mathcal{C}}$ with $r \gg 0$, we define $\mathcal{C}_r := \{u \in \mathcal{C} : r \geq u \geq 0\}$ and $\bar{\mathcal{C}}_r := \{u \in \bar{\mathcal{C}} : r \geq u \geq 0\}$.

In this paper, we always equip $\bar{\mathcal{C}}$ with the maximum norm $\|\cdot\|$ and the positive cone $\bar{\mathcal{C}}_+ = \{\phi \in \bar{\mathcal{C}} : \phi(\theta) \geq 0\ \forall \theta \in [-\tau, 0]\}$ so that $\bar{\mathcal{C}}$ is an ordered Banach space. We also equip $\mathcal{C}$ with the compact open topology, that is, $v^n \to v$ in $\mathcal{C}$ means that the sequence of functions $v^n(\theta, x)$ converges to $v(\theta, x)$ uniformly for (θ, x) in every compact set. Moreover, we can define the metric function $d(\cdot, \cdot)$ in $\mathcal{C}$ with respect to this topology by

$$d(u, v) = \sum_{k=0}^{\infty} \frac{\max_{|x| \leq k, \theta \in [-\tau, 0]} |u(\theta, x) - v(\theta, x)|}{2^k} \quad \forall u, v \in \mathcal{C}$$

so that $(\mathcal{C}, d)$ is a metric space.

Define the reflection operator $\mathcal{R}$ by $\mathcal{R}[u](\theta, x) = u(\theta, -x)$. Given $y \in \mathcal{H}$, define the translation operator T_y by $T_y[u](\theta, x) = u(\theta, x - y)$.

Let $\beta \in \bar{\mathcal{C}}$ with $\beta \gg 0$ and $Q = (Q_1, \dots, Q_k) : \mathcal{C}_\beta \to \mathcal{C}_\beta$. We impose the following hypotheses on Q:

(A1) $Q[\mathcal{R}[u]] = \mathcal{R}[Q[u]]$, $T_y[Q[u]] = Q[T_y[u]]\ \forall y \in \mathcal{H}$.
(A2) $Q : \mathcal{C}_\beta \to \mathcal{C}_\beta$ is continuous with respect to the compact open topology.
(A3) One of the following two properties holds:
  (a) $\{Q[u](\cdot, x) : u \in \mathcal{C}_\beta,\ x \in \mathcal{H}\}$ is a precompact subset of $\bar{\mathcal{C}}$.
  (b) There exists a nonnegative number $\varsigma < \tau$ such that $Q[u](\theta, x) = u(\theta + \varsigma, x)$ for $-\tau \leq \theta < -\varsigma$, the operator

$$S[u](\theta, x) := \begin{cases} u(0, x), & -\tau \leq \theta < -\varsigma, \\ Q[u](\theta, x), & -\varsigma \leq \theta \leq 0, \end{cases}$$

  is continuous on $\mathcal{C}_\beta$, and $\{S[u](\cdot, x) : u \in \mathcal{C}_\beta, x \in \mathcal{H}\}$ is a precompact subset of $\bar{\mathcal{C}}$.

(A4) $Q : \mathcal{C}_\beta \to \mathcal{C}_\beta$ is monotone (order preserving) in the sense that $Q[u] \geq Q[v]$ whenever $u \geq v$ in $\mathcal{C}_\beta$.

By the Arzela-Ascoli theorem, it is easy to see that (A3)(a) is equivalent to the statement that $\{Q[u](\cdot, x) : u \in \mathcal{C}_\beta, x \in \mathcal{H}\}$ is a family of equicontinuous functions of $\theta \in [-\tau, 0]$. Similarly, if (A3)(b) holds, then $\{S[u](\cdot, x) : u \in \mathcal{C}_\beta, x \in \mathcal{H}\}$ is a family of equicontinuous functions of $\theta \in [-\tau, 0]$. Note that hypothesis (A1) implies that $Q[v] \in \bar{\mathcal{C}}_\beta$ whenever $v \in \bar{\mathcal{C}}_\beta$. Thus, Q is also a map from $\bar{\mathcal{C}}_\beta$ to $\bar{\mathcal{C}}_\beta$.

(A5) $Q : \bar{\mathcal{C}}_\beta \to \bar{\mathcal{C}}_\beta$ admits exactly two fixed points 0 and β, and for any positive number ϵ, there is $\alpha \in \bar{\mathcal{C}}_\beta$ with $\|\alpha\| < \epsilon$ such that $Q[\alpha] \gg \alpha$.

Clearly, hypotheses (A3) and (A5) imply that for any $\gamma \in \bar{\mathcal{C}}_\beta$ with $0 \ll \gamma \ll \beta$, $Q^n[\gamma] \to \beta$ as $n \to +\infty$. We remark that hypothesis (A3) is motivated by time-delayed reaction-diffusion systems. For such a system, let Q_ς be the solution map at time ς. If ς is less than the delay τ, then Q_ς satisfies property (A3)(b) (see, e.g., [16, sec. 3.6]).

Throughout this paper, we assume that Q satisfies hypotheses (A1)–(A5). Let $\tilde{\mathcal{C}}$ be the set of all continuous functions from $[-\tau, 0] \times \mathbb{R}$ to $\mathbb{R}^k$. In the case where $\mathcal{H} = \mathbb{Z}$, we define an operator $\tilde{Q}$ on the set $\tilde{\mathcal{C}}_\beta$ by

$$\tilde{Q}[v](\theta, s) := Q[v(\cdot, \cdot + s)](\theta, 0) \quad \forall \theta \in [-\tau, 0],\ s \in \mathbb{R}.$$

It is easy to see that $\tilde{Q}$ satisfies hypotheses (A1), (A3), (A4), and (A5) with $\mathcal{H} = \mathbb{R}$. The following lemma shows that $\tilde{Q}$ also satisfies (A2):

LEMMA 2.1 *$\tilde{Q}$ is continuous on $\tilde{\mathcal{C}}_\beta$ with respect to the compact open topology.*

PROOF: Given $v \in \tilde{\mathcal{C}}_\beta$. For any $s \in \mathbb{R}$, we define $v_s \in \mathcal{C}_\beta$ by $v_s(\theta, x) = v(\theta, x + s)$ for $\theta \in [-\tau, 0]$, $x \in \mathcal{H}$. We first prove the following claim:

Claim. Let $[a, b]$ be a given bounded interval in $\mathbb{R}$. For any $\epsilon > 0$, there exist $\delta = \delta(\epsilon) > 0$ and $N = N(\epsilon) > 0$ such that if for some $s \in [a, b]$, $|u(\theta, x) - v_s(\theta, x)| < \delta\ \forall x \in [-N, N]_{\mathcal{H}}$, $\theta \in [-\tau, 0]$, then we have $|Q[u](\theta, 0) - Q[v_s](\theta, 0)| < \epsilon$ $\forall \theta \in [-\tau, 0]$, where $[-N, N]_{\mathcal{H}} = \{x \in \mathcal{H} : -N \leq x \leq N\}$.

Indeed, for any $s_0 \in [a, b]$, since Q is continuous at v_{s_0}, there exist $\delta_{s_0} > 0$ and $N_{s_0} > 0$ such that

$$|Q[u](\theta, 0) - Q[v_{s_0}](\theta, 0)| < \frac{\epsilon}{2}$$

provided $|u(\theta, x) - v_{s_0}(\theta, x)| < \delta_{s_0}\ \forall x \in [-N_{s_0}, N_{s_0}]_{\mathcal{H}}$. It is easy to see that v_s is a continuous map from $\mathbb{R}$ to $\mathcal{C}_\beta$. Thus, there exists $m_{s_0} > 0$ such that

$$|v_s(\theta, x) - v_{s_0}(\theta, x)| < \frac{\delta_{s_0}}{2} \quad \forall x \in [-N_{s_0}, N_{s_0}]_{\mathcal{H}},\ \theta \in [-\tau, 0],$$

provided that $|s - s_0| < m_{s_0}$. It then follows that

$$|Q[v_s](\theta, 0) - Q[v_{s_0}](\theta, 0)| < \frac{\epsilon}{2} \quad \forall \theta \in [-\tau, 0]$$

provided that $|s - s_0| < m_{s_0}$. By the compactness of $[a, b]$, there exists a finite sequence $\{s_1, \ldots, s_k\}$ such that $[a, b] \subset \bigcup_{i=1}^k B(s_i, m_{s_i})$. Let $\delta = \min\{\delta_{s_i}/2 : 1 \leq i \leq k\}$ and $N = \max\{N_{s_i} : 1 \leq i \leq k\}$. Assume that for some $s \in [a, b]$, $|u(\theta, x) - v_s(\theta, x)| < \delta\ \forall x \in [-N, N]_{\mathcal{H}}$, $\theta \in [-\tau, 0]$. Then $s \in B(s_i, m_{s_i})$ for some i, and hence

$$|Q[v_s](\theta, 0) - Q[v_{s_i}](\theta, 0)| < \frac{\epsilon}{2}.$$

Since

$$\begin{aligned}|u(\theta, x) - v_{s_i}(\theta, x)| &\leq |u(\theta, x) - v_s(\theta, x)| + |v_s(\theta, x) - v_{s_i}(\theta, x)| \\ &< \delta + \frac{\delta_{s_i}}{2} \leq \delta_{s_i}\end{aligned}$$

for all $x \in [-N_{s_i}, N_{s_i}]_{\mathcal{H}}$, $\theta \in [-\tau, 0]$, we have

$$|Q[u](\theta, 0) - Q[v_{s_i}](\theta, 0)| < \frac{\epsilon}{2}.$$

Thus, we obtain

$$\begin{aligned}|Q[u](\theta, 0) - Q[v_s](\theta, 0)| \leq{}& |Q[u](\theta, 0) - Q[v_{s_i}](\theta, 0)| \\ &+ |Q[v_{s_i}](\theta, 0) - Q[v_s](\theta, 0)| \\ <{}& \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

This proves the claim above.

Let $v^n \to v$ in $\tilde{\mathcal{C}}_\beta$. Given a bounded interval $[a, b] \subset \mathbb{R}$ and $\epsilon > 0$, let δ and N be defined as in the above claim. Since $\lim_{n\to\infty} v^n(\theta, x + s) = v(\theta, x + s)$ uniformly for $x \in [-N, N]_{\mathcal{H}}$, $\theta \in [-\tau, 0]$, and $s \in [a, b]$, there exists $n_0 = n_0(\epsilon) > 0$ such that for all $n \geq n_0$, we have

$$|v_s^n(\theta, x) - v_s(\theta, x)| < \delta \quad \forall x \in [-N, N]_{\mathcal{H}},\ \theta \in [-\tau, 0],\ s \in [a, b].$$

By the claim above, it follows that for all $n \geq n_0$, we have

$$|\tilde{Q}[v^n](\theta, s) - \tilde{Q}[v](\theta, s)| = |Q[v_s^n](\theta, 0) - Q[v_s](\theta, 0)| < \epsilon$$

for all $\theta \in [-\tau, 0]$ and $s \in [a, b]$. This implies that $\tilde{Q}[v^n](\theta, s)$ converges to $\tilde{Q}[v](\theta, s)$ uniformly for $\theta \in [-\tau, 0]$, $s \in [a, b]$. Consequently, $\tilde{Q}[v^n]$ converges to $\tilde{Q}[v]$ with respect to the compact open topology. □

Remark 2.2. In Lemma 2.1, $\mathbb{R}$ can be replaced by any set $B \subset \mathbb{R}$ such that $\mathcal{H} \subset B$ and $x - y, x + y \in B$ whenever $x, y \in B$. Moreover, for any $v = v(\theta, s)$, $\theta \in [-\tau, 0]$, and $s \in B$, we can also define the extension $\tilde{Q}$ of Q provided v is continuous in θ.

In view of Lemma 2.1, we assume, without loss of generality, that $\mathcal{H} = \mathbb{R}$ in the rest of this section. We start with the discrete-time semiflow on $\mathcal{C}_\beta$:

$$u_{n+1} = Q[u_n], \quad n \geq 0,\ u_0 \in \mathcal{C}_\beta.$$

By an induction argument, it is easy to prove the following comparison principle (see, e.g., [21, prop. 2.1]).

PROPOSITION 2.3 *Let R_1 or R_2 be an order-preserving operator. Suppose the sequence $\{v_n\}$ satisfies $v_{n+1} \geq R_1[v_n]$ and the sequence $\{w_n\}$ satisfies $w_{n+1} \leq R_2[w_n]$ for all n. Suppose also that $R_1[u] \geq R_2[u]$ for all functions u and that $v_0 \geq w_0$. Then $v_n \geq w_n$ for all n.*

Let $\alpha \in \bar{\mathcal{C}}_\beta$ with $0 \ll \alpha \ll \beta$, and assume that $\phi = (\phi^1, \ldots, \phi^k) \in \mathcal{C}_\beta$ has the following properties:

(B1) $\phi^i(\theta, \cdot)$ is a nonincreasing function for any fixed $\theta \in [-\tau, 0]$ and $1 \leq i \leq k$.
(B2) $\phi^i(\theta, x) = 0$ for any $\theta \in [-\tau, 0]$, $x \geq 0$, and $1 \leq i \leq k$.
(B3) $\phi(\theta, -\infty) = \alpha(\theta)$ for any $\theta \in [-\tau, 0]$.

Then we have the following result:

LEMMA 2.4 *$\{\phi(\cdot, x) : x \in \mathcal{H}\}$ is a family of equicontinuous functions of $\theta \in [-\tau, 0]$.*

PROOF: Define $\psi(\theta, \eta) = \phi(\theta, \tan\eta)$. Then ψ is a continuous function on $[-\tau, 0] \times (-\frac{\pi}{2}, \frac{\pi}{2})$ and is nonincreasing in $\eta \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Since ϕ satisfies (B2) and (B3), Dini's theorem implies that ψ has a natural continuous extension to the compact set $[-\tau, 0] \times [-\frac{\pi}{2}, \frac{\pi}{2}]$. Thus, the equicontinuity of $\{\phi(\cdot, x) : x \in \mathcal{H}\} = \{\psi(\cdot, \eta) : \eta \in (-\frac{\pi}{2}, \frac{\pi}{2})\}$ follows from the uniform continuity of ψ on $[-\tau, 0] \times [-\frac{\pi}{2}, \frac{\pi}{2}]$. □

Given a real number c, we define the operator $R_c = (R_c^1, \ldots, R_c^k)$ by

$$R_c[a](\theta, s) = \max\{\phi(\theta, s), T_{-c}[Q[a]](\theta, s)\}$$

and a sequence of vector-valued functions $a_n(c; \theta, s)$ of $(\theta, s) \in [-\tau, 0] \times \mathbb{R}$ by the recursion

$$a_0(c; \theta, s) = \phi(\theta, s), \quad a_{n+1}(c; \theta, s) = R_c[a_n(c; \cdot)](\theta, s). \tag{2.1}$$

Before we prove the main results in this section, we need a series of lemmas.

LEMMA 2.5 *The following statements are valid*:

(i) *R_c is order preserving.*
(ii) *$a_n(c; \theta, s)$ is between 0 and β, nondecreasing in n, nonincreasing in s and c, and continuous in (c, s, θ).*
(iii) *$a_n(c; \cdot, -\infty)$ exists, $a_n(c; \cdot, -\infty) \geq Q^n[\alpha]$, and $a_n(c; \cdot, \infty) = 0$ for each n.*
(iv) *$\lim_{n\to\infty} a_n(c; \theta, s) = a(c; \theta, s)$ exists and the limit is uniform in θ, a is nonincreasing in s and c, and $a(c; \cdot, -\infty) = \beta(\cdot)$.*

PROOF: Statements (i), (ii), and (iii) are obvious. We prove only (iv). First, we claim that $\{a_n(c;\cdot,s) : n \geq 0,\ c, s \in \mathbb{R}\}$ is a family of equicontinuous functions. If (A3)(a) holds, the claim is obvious. Now consider the case where (A3)(b) holds. In fact, we have shown that $\{a_0(\cdot,s) = \phi(\cdot,s) : s \in \mathbb{R}\}$ is a family of equicontinuous functions. This means that given any $\epsilon > 0$, there is some $\delta_0 > 0$ such that if $|\theta_1 - \theta_2| < \delta_0$, then $|a_0(\theta_1,s) - a_0(\theta_2,s)| < \epsilon$ for any $s \in \mathbb{R}$. Consider $Q[a_0]$. By hypothesis (A3), $\{S[\mathcal{C}_\beta](\theta,x) : x \in \mathbb{R}\}$ is family of equicontinuous functions in $\theta \in [-\tau,0]$; that is, there is $\delta > 0$ such that for any $v \in \mathcal{C}_\beta$, $s \in \mathbb{R}$, and $\theta_1, \theta_2 \in [-\tau,0]$ with $|\theta_1 - \theta_2| < \delta$, we have $|S[v](\theta_1,s) - S[v](\theta_2,s)| < \epsilon$.

We first consider

$$Q[a_0](\theta,x) = \begin{cases} a_0(\theta+\varsigma,x), & -\tau \leq \theta < -\varsigma, \\ S[a_0](\theta,x), & -\varsigma \leq \theta \leq 0. \end{cases}$$

It follows that for any $x \in \mathbb{R}$, $Q[a_0](\theta_1,x) - Q[a_0](\theta_2,x) < \epsilon$ whenever $-\varsigma \leq \theta_1, \theta_2 \leq 0$ and $|\theta_1 - \theta_2| < \delta$ or $-\tau \leq \theta_1, \theta_2 \leq -\varsigma$ and $|\theta_1 - \theta_2| < \delta_0$. Since

$$a_1 = \max\{a_0, T_{-c}[Q[a_0]]\},$$

we have $|a_1(\theta_1,x) - a_1(\theta_2,x)| < \epsilon$ whenever $-\varsigma \leq \theta_1, \theta_2 \leq 0$ and $|\theta_1 - \theta_2| < \delta_1 := \min\{\delta, \delta_0\}$ or $-\tau \leq \theta_1, \theta_2 \leq -\varsigma$ and $|\theta_1 - \theta_2| < \delta_0$. This implies that $|a_1(\theta_1,x) - a_2(\theta_2,x)| < 2\epsilon$ whenever $-\tau \leq \theta_1, \theta_2 \leq 0$ and $|\theta_1 - \theta_2| < \min\{\delta_1, \delta_0\} = \delta_1$.

Next we consider

$$Q[a_1](\theta,x) = \begin{cases} a_1(\theta+\varsigma,x), & -\tau \leq \theta < -\varsigma, \\ S[a_1](\theta,x), & -\varsigma \leq \theta \leq 0. \end{cases}$$

It follows that $Q[a_1](\theta_1,x) - Q[a_1](\theta_2,x) < \epsilon$ whenever $-\varsigma \leq \theta_1, \theta_2 \leq 0$ and $|\theta_1 - \theta_2| < \delta$, or $-2\varsigma \leq \theta_1, \theta_2 \leq -\varsigma$ and $|\theta_1 - \theta_2| < \delta_1$, or $-\tau \leq \theta_1, \theta_2 \leq -2\varsigma$ and $|\theta_1 - \theta_2| < \delta_0$. Since $a_2 = \max\{a_0, T_{-c}[Q[a_1]]\}$, we see that $|a_2(\theta_1,x) - a_2(\theta_2,x)| < \epsilon$ whenever $-\varsigma \leq \theta_1, \theta_2 \leq 0$ and $|\theta_1 - \theta_2| < \delta_1$, or $-2\varsigma \leq \theta_1, \theta_2 \leq -\varsigma$ and $|\theta_1 - \theta_2| < \min\{\delta_1, \delta_0\} = \delta_1$, or $-\tau \leq \theta_1, \theta_2 \leq -2\varsigma$ and $|\theta_1 - \theta_2| < \delta_0$. It then follows that $|a_2(\theta_1,x) - a_2(\theta_2,x)| < 2\epsilon$ whenever $-\tau \leq \theta_1, \theta_2 \leq 0$ and $|\theta_1 - \theta_2| < \min\{\delta_1, \delta_0\} = \delta_1$.

Repeating this procedure, we can show that for any $x \in \mathbb{R}$ and $n \geq 1$, $|a_n(\theta_1,x) - a_n(\theta_2,x)| < 2\epsilon$ whenever $-\tau \leq \theta_1, \theta_2 \leq 0$ and $|\theta_1 - \theta_2| < \delta_1$. This proves our claim, and hence statement (iv). □

LEMMA 2.6 *$a(c;\cdot,\infty) = \beta$ if and only if there is some n such that $a_n(c;\cdot,0) \gg \phi(\cdot,-\infty) = \alpha$.*

PROOF: The only if part is obvious since a_n increases to a, which is identically β. This implies that $a_n(c;\cdot,0)$ increases to β uniformly. Since $\beta \gg \alpha$, we have $a_n(c;\cdot,0) \gg \phi(\cdot,-\infty) = \alpha$.

Now consider the if part. Clearly, $a_n(c; \cdot, 0) \gg \alpha$ implies that for sufficiently small $t > 0$, $a_n(c; \cdot, t) \gg \alpha$. Thus, $T_{-t}[a_n(c; \cdot)] \geq a_0(c; \cdot)$. We claim that $T_{-t}[a_{n+i}(c; \cdot)] \geq a_i(c; \cdot)$ for all $i \geq 1$. Note that

$$\begin{cases} T_{-t}[a_{n+1}(c; \cdot)] \geq T_{-t}[a_n(c; \cdot)] \geq a_0(c; \cdot) \\ T_{-t}[a_{n+1}(c; \cdot)] \geq T_{-c-t}[Q[a_n(c; \cdot)]] \geq T_{-c}[Q[a_0(c; \cdot)]]. \end{cases}$$

Thus, our claim holds for $i = 1$. By an induction argument, we can prove that the claim holds for all i. In other words, we have $a_{n+i}(c; \cdot, s+t) \geq a_i(c; \cdot, s)\ \forall s \in \mathcal{H}$. Letting $i \to \infty$, we obtain $a(c; \cdot, s+t) \geq a(c; \cdot, s)\ \forall s \in \mathcal{H}$, which implies that $a(c; \cdot, s) \equiv a(c; \cdot, -\infty) = \beta$, and hence $a(c; \cdot, \infty) = \beta$. □

Define

$$c^* := \sup\{c : a(c; \cdot, \infty) = \beta\}. \tag{2.2}$$

It is easy to show that $c^* > -\infty$ by Lemma 2.6, but c^* may be infinity. Moreover, if $a(c_0; \cdot, \infty) = \beta$ for some c_0, then $a(c_0; \cdot, 0) = \beta \gg \alpha$. This implies that there is some n such that $a_n(c_0; \cdot, 0) \gg \alpha$. Since a_n is continuous in c in a neighborhood of c_0, $a_n(c; \cdot, 0) \gg \alpha$. Hence, we have the following result:

LEMMA 2.7 *$a(c; \theta, s) \equiv \beta(\theta)$ if and only if $c < c^*$.*

LEMMA 2.8 *Let $\hat{\alpha} \in \bar{\mathcal{C}}_\beta$ with $0 \ll \hat{\alpha} \ll \beta$, and let $\hat{\phi}$ satisfy* (B1)–(B3) *with α replaced by $\hat{\alpha}$. Define $\hat{a}_n$ recursively by* (2.1) *with ϕ replaced by $\hat{\phi}$. Denote $\hat{a} = \lim_{n\to\infty} \hat{a}_n$. Then $\hat{a}(c; \cdot, \infty) = a(c; \cdot, \infty)$.*

PROOF: Since $\beta \gg \alpha$ and $Q^n[\hat{\alpha}] \to \beta$ as $n \to \infty$, there exists n_0 such that $Q^n[\hat{\alpha}] \gg \alpha$ for $n \geq n_0$. From Lemma 2.5(iii), there exists $t = t(c) > 0$ such that $\hat{a}_n(c; \cdot, -t) \gg \alpha$ for $n \geq n_0$. Since $\hat{a}_n$ and ϕ are nonincreasing in s, we have

$$T_t[\hat{a}_n(c; \cdot)] \geq a_0(c; \cdot), \quad n \geq n_0. \tag{2.3}$$

Consider the sequence $T_t[\hat{a}_{n_0+l}(c; \cdot)]$ for $l \geq 0$. We claim that

$$T_t[\hat{a}_{n_0+l+1}(c; \cdot)] \geq T_t[R_c[\hat{a}_{n_0}(c; \cdot)]], \quad l \geq 0. \tag{2.4}$$

Indeed, we observe that because of (2.3), the right-hand side of inequality (2.4) is not greater than

$$\max\{T_t[\hat{a}_{n_0}(c; \cdot)], T_{t-c}[Q[\hat{a}_{n_0+l}(c; \cdot)]]\},$$

which in turn is not greater than $T_t[\hat{a}_{n_0+l+1}(c; \cdot)]$. From our claim and Proposition 2.3, we have

$$T_t[\hat{a}_{n_0+l}(c; \cdot)] \geq a_l(c; \cdot) \quad \forall l \geq 0.$$

Letting $l \to \infty$ and then $s \to \infty$, we have $\hat{a}(c; \cdot, \infty) \geq a(c; \cdot, \infty)$. Exchanging the positions of ϕ and $\hat{\phi}$ and repeating the proof above, we obtain the opposite inequality. This completes the proof. □

By the definition of c^*, we can obtain the following lemma easily:

LEMMA 2.9 *Let* $0 \ll \beta_1 \le \beta_2$ *in* $\bar{\mathcal{C}}$, Q_i *satisfy* (A1)–(A5) *with* β *replaced by* β_i, *and* c_i^* *be defined as in* (2.2) *for* Q_i, $i = 1, 2$. *If* $Q_1[u] \le Q_2[u]$ *for all* $u \in \mathcal{C}_{\beta_1}$, *then* $c_1^* \le c_2^*$.

LEMMA 2.10 $a(c; \cdot, \infty)$ *is continuous,* $Q[a(c; \cdot, \infty)] = a(c; \cdot, \infty)$, *and* $a(c; \cdot, \infty) = 0$ *for* $c \ge c^*$.

PROOF: By Lemma 2.5(iv), $\{a(c; \cdot, s) : s \in \mathcal{H}\}$ is equicontinuous. Then

$$a(c; \cdot, \infty) = \lim_{s \to \infty} a(c; \cdot, s)$$

is continuous, and $\lim_{t \to \infty} T_{-t}[a] = a(c; \cdot, \infty)$ with respect to the compact open topology.

Note that $a_n(c; \theta, s) \le a(c; \theta, s)$. Let $\tilde{a}(\theta, s)$ be a continuous function on $[-\tau, 0] \times \mathcal{H}$ that is nonincreasing in s. Moreover, suppose $\tilde{a}(\cdot, \infty) = a(c; \cdot, \infty)$ and $\tilde{a}(\theta, s) \ge a(c; \theta, s)$. Then $a_n(c; \theta, s) \le \tilde{a}(\theta, s)$. For any $s > 0$, we have

$$a_{n+1}(c; \theta, s) \le Q[T_{-c}[\tilde{a}]](\theta, s).$$

Letting $n \to \infty$, we then obtain $a(c; \theta, s) \le Q[T_{-c}[\tilde{a}]](\theta, s)$ and

$$a(c; \theta, \infty) \le \lim_{s \to \infty} Q[T_{-c}[\tilde{a}]](\theta, s) = Q[a(c; \cdot, \infty)](\theta) \quad \forall \theta \in [-\tau, 0].$$

Assume, for the sake of contradiction, that $Q[a(c; \cdot, \infty)] > a(c; \cdot, \infty)$. Then there exist i_0 with $1 \le i_0 \le k$ and $\theta_0 \in [-\tau, 0]$ such that $Q_{i_0}[a(c; \cdot, \infty)](\theta_0) > a^{i_0}(c; \theta_0, \infty)$. Denote by S_i the support of $a^i(c; \theta, \infty)$ and by $\dot{S}_i$ the interior of S_i in $[-\tau, 0]$ for $1 \le i \le k$. By continuity, there are a compact set $S_i' \subset \dot{S}_i$ and a vector-valued function $\delta \in \bar{\mathcal{C}}$ with S_i' being the support of the i^{th} component of δ such that

(a) $0 < \delta < a(c; \cdot, \infty)$,
(b) if $S_i \ne \varnothing$, then $\delta^i(\theta) < a^i(c; \theta, \infty)$ on S_i', and
(c) $a^{i_0}(c; \theta_0, \infty) < Q_{i_0}[\delta](\theta_0)$.

For each positive integer l, $a(c; \cdot, l) \ge a(c; \cdot, \infty) \ge \delta$ and $a_n(c; \theta, l) \to a(c; \theta, l)$ uniformly for $\theta \in [-\tau, 0]$ as $n \to \infty$. Since $a^i(c; \theta, l) \ge a^i(c; \theta, \infty) > \delta^i(\theta)$ $\forall \theta \in S_i'$, $1 \le i \le k$, we can choose a sufficiently small $\epsilon > 0$ such that $a^i(c; \theta, l) > \delta^i(\theta) + \epsilon$ $\forall \theta \in S_i'$, $1 \le i \le k$. Note that $a_n(c; \theta, l) \to a(c; \theta, l)$ uniformly for $\theta \in [-\tau, 0]$. It follows that there is some n_l such that $a_{n_l}^i(c; \theta, l) \ge \delta^i(\theta)$ on S_i' for any $1 \le i \le k$. Clearly, $a_{n_l}^i(c; \theta, l) \ge \delta^i(\theta) = 0$ on $[-\tau, 0] \setminus S_i'$. Thus, we have $a_{n_l}(c; \theta, l) \ge \delta(\theta)$ $\forall \theta \in [-\tau, 0]$.

Let $\psi(\theta, s)$ be a continuous, nonincreasing-in-s, vector-valued function such that $\psi(\cdot, s) = \delta(\cdot)$ for $s \le -1$ and $\psi(\cdot, s) = 0$ for $s \ge 0$. Then $a_{n_l}(c; \theta, s) \ge \psi(\theta, s - l)$, and hence

$$a(c; \theta, s) \ge a_{n_l+1}(c; \theta, s) \ge Q[T_{l-c}[\psi]](\theta, s).$$

Letting $l \to \infty$ and then $s \to \infty$, we have $a(c; \cdot, \infty) \ge Q[\delta]$, which contradicts statement (c). Thus, $Q[a(c; \cdot, \infty)] = a(c; \cdot, \infty)$. Since $a(c; \cdot, \infty) < \beta$ for $c \ge c^*$, we obtain $a(c; \cdot, \infty) = 0$ by hypothesis (A5). □

THEOREM 2.11 *Let* $u_0 \in \mathcal{C}_\beta$ *be such that* $0 \le u_0 \ll \beta$ *and* $u_0(\cdot, x) = 0$ *for* x *outside a bounded interval, and let* $u_n = Q[u_{n-1}]$ *for* $n \ge 1$. *Then for any* $c > c^*$, *there holds* $\lim_{n\to\infty, |x|\ge nc} u_n(\theta, x) = 0$ *uniformly for* $\theta \in [-\tau, 0]$.

PROOF: Suppose that $u_0(\cdot, x) = 0$ if $x \ge \rho - 1$. Moreover, without loss of generality, assume that $u_0(\cdot, x) \ll \alpha$ where α is defined in (A5). Let $\phi(\theta, s)$ be a continuous and nonincreasing-in-s vector-valued function such that $\phi(\theta, s) = \alpha(\theta)$ for $s \le -1$, and $\phi(\theta, s) = 0$ for $s \ge 0$. We define a_n and c^* as in (2.1) and (2.2). Let

$$v_n(\theta, x) = a_n(c^*; \theta, x - nc^* - \rho).$$

Then

$$u_0(\theta, x) \le \phi(\theta, x - \rho) = v_0(\theta, x).$$

By the definition of a_n, we see that $v_{n+1} \ge Q[v_n]$ for all n. Hence, $u_n \le v_n$ by Proposition 2.3. If $x > nc$, then

$$\begin{aligned} u_n(\theta, x) \le a_n(c^*; \theta, x - nc^* - \rho) &\le a_n(c^*; \theta, nc - nc^* - \rho) \\ &\le a(c^*; \theta, nc - nc^* - \rho). \end{aligned}$$

This implies that $\lim_{n\to\infty, x\ge nc} u_n(\theta, x) = 0$ uniformly for $\theta \in [-\tau, 0]$.

Let $\tilde{u}_0 = \mathcal{R}[u_0]$. Then we have

$$\tilde{u}_n = Q^n[\tilde{u}_0] = Q^n[\mathcal{R}[u_0]] = \mathcal{R}[Q^n[u_0]].$$

By a similar argument, it follows that $\lim_{n\to\infty, x\ge nc} \tilde{u}_n(\theta, x) = 0$, and hence $\lim_{n\to\infty, x\le -nc} u_n(\theta, x) = 0$ uniformly for $\theta \in [-\tau, 0]$. □

Let $\varpi(\cdot)\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a smooth, nonincreasing function such that

$$\varpi(s) = \begin{cases} 1, & s \le \frac{1}{2}, \\ 0, & s \ge 1. \end{cases} \tag{2.5}$$

For any real number $B > 0$, we define the map Q_B on $\mathcal{C}$ by

$$Q_B[u](\theta, x) = Q\left[\varpi\left(\frac{|\cdot|}{B}\right) T_{-x}[u]\right](\theta, 0) \quad \forall (\theta, x) \in [-\tau, 0] \times \mathcal{H}.$$

Then we have the following lemma about Q_B.

LEMMA 2.12 *The following statements hold*:

(i) Q_B *satisfies hypotheses* (A2)–(A4), $Q_B[0] = 0$, *and* $Q_B\mathcal{R} = \mathcal{R}Q_B$, $T_y Q_B = Q_B T_y$ *for any* $y \in \mathcal{H}$.

(ii) *For each* u, $Q_B[u]$ *is nondecreasing in* B *and converges to* $Q[u]$ *as* $B \to \infty$.

(iii) $Q_B[u](\theta, x_0)$ *depends only on the values of* u *in the set* $[-\tau, 0] \times [x_0 - B, x_0 + B]$.

LEMMA 2.13 *For any* $\epsilon \in \bar{\mathcal{C}}_\beta$ *with* $\epsilon \gg 0$, *there is* B *such that* $Q_B[\alpha] \gg \alpha$ *and* $\lim_{n\to\infty} Q_B^n[\alpha] \to \beta_B \gg \beta - \epsilon$, *where* α *is defined as in hypothesis* (A5).

PROOF: By conclusion (ii) of Lemma 2.12, there is $B_0 > 0$ such that $Q_B[\alpha] \geq \alpha$ for $B \geq B_0$ since $Q[\alpha] \gg \alpha$. Since $Q^n[\alpha] \to \beta$, there is some n_0 such that $Q^{n_0}[\alpha] \gg \beta - \epsilon$, and hence there is some B_0' such that $Q_B^{n_0}[\alpha] \gg \beta - \epsilon$ for $B \geq B_0'$. Choose $B \geq \max\{B_0, B_0'\}$. Then $Q_B^n[\alpha] \geq Q_B^{n-1}[\alpha]$ and $Q_B^n[\alpha] \to \beta_B \geq Q_B^{n_0}[\alpha] \gg \beta - \epsilon$. □

Let ϕ satisfy (B1)–(B3) with $\phi(\cdot, s) = \alpha(\cdot)$ for $s \leq -1$. Define

$$\tilde{R}_c[a](\theta, s) = \max\{\phi(\theta, s), T_{-c}[Q_B[a]](\theta, s)\}$$

and

$$\tilde{a}_0(c; \theta, s) = \phi(\theta, s), \quad \tilde{a}_{n+1}(c; \theta, s) = \tilde{R}_c[a_n(c, \cdot)](\theta, s).$$

As argued in Lemma 2.5, we see that $\tilde{a}_n(c, \theta, s)$ is between 0 and β_B, nondecreasing in n, nonincreasing in s and c, and continuous in c, s, and θ. Moreover,

$$\tilde{a}_n(c; \theta, s) = \begin{cases} Q_B^n[\alpha](\theta), & s \leq -1 - n(B + c), \\ 0, & s \geq n(B - c). \end{cases} \tag{2.6}$$

Then $\tilde{a}_n(c; \cdot, -\infty) = Q_B^n[\alpha]$. Fix $\bar{c} \in (c, c^*)$. Note that the sequence $a_n(\bar{c}; \theta, s) = R_{\bar{c}}^n[\phi](\theta, s) \to \beta(\theta)$. By Lemma 2.6, there is an integer N such that $a_N(\bar{c}; \cdot, 0) \gg \alpha$. Furthermore, we can choose B so large that $\tilde{a}_N(\bar{c}; \cdot, 0) \gg \alpha$ also. Therefore,

$$\tilde{a}_{n+1}(\bar{c}; \theta, s) = Q_B[T_{-\bar{c}}[\tilde{a}_n(\bar{c}; \cdot)]](\theta, s) \quad \forall n \geq N.$$

Define the sequence e_n by

$$e_n(\theta, x) = \tilde{a}_m(\bar{c}; \theta, |x| - (n + A)\bar{c}), \quad n > 0,$$

where $m > N$, $A > (1/\bar{c})(1 + m(B + \bar{c}) + 2B)$. By the definition of e_n we have

$$e_n(\theta, x) = \begin{cases} Q_B^m[\alpha](\theta), & |x| \leq (n + A)\bar{c} - 1 - m(B + \bar{c}), \\ 0, & |x| \geq (n + A)\bar{c} + m(B - \bar{c}). \end{cases} \tag{2.7}$$

LEMMA 2.14 *$e_{n+1} \leq Q_B[e_n]$ for $n \geq 0$.*

PROOF: For any $x_0 \in \mathcal{H}$, if $|x_0| \leq (n + A)\bar{c} - 1 - m(B + \bar{c}) - B$, then for any x with $|x - x_0| \leq B$, $x \leq (n + A)\bar{c} - 1 - m(B + \bar{c})$ and then $e_n(\theta, x) = Q_B^m[\alpha](\theta)$. This implies that $Q[e_n](\theta, x_0) = Q_B^{m+1}[\alpha] \geq Q_B^m[\alpha] = e_n(\theta, x_0)$.

Now suppose that $|x_0| > (n + A)\bar{c} - 1 - m(B + \bar{c}) - B$. Let $x_0 \geq 0$. Since $A > (1/\bar{c})(1 + m(B + \bar{c}) + 2B)$, we see that $x > 0$ for any x with $|x - x_0| \leq B$. Then

$$e_n(\theta, x) = \tilde{a}_m(\bar{c}; \theta, x - (n + A)\bar{c}) \geq \tilde{a}_{m-1}(\bar{c}; \theta, x - (n + A)\bar{c}).$$

It follows that

$$\begin{aligned} Q[e_n](\theta, x_0) &= Q[\tilde{a}_m](\theta, x_0 - (n + A)\bar{c}) \\ &\geq Q[\tilde{a}_{m-1}](\theta, x_0 - (n + A)\bar{c}) \\ &= \tilde{a}_m(\bar{c}; \theta, x_0 - (n + A)\bar{c} - \bar{c}) \\ &= e_{n+1}(\theta, x_0). \end{aligned}$$

The case where $x_0 < 0$ can be proved in a similar way. □

THEOREM 2.15 *For any $c < c^*$ and any $\sigma \in \bar{\mathcal{C}}_\beta$ with $\sigma \gg 0$, there exists $r_\sigma > 0$ such that if $u_0(\cdot, x) \geq \sigma(\cdot)$ for x on an interval of length $2r_\sigma$, then $\lim_{n\to\infty, |x|\leq nc} u_n(\theta, x) = \beta(\theta)$ uniformly for $\theta \in [-\tau, 0]$.*

PROOF: Without loss of generality, we assume that the interval of length $2r_\sigma$ is $[-r_\sigma, r_\sigma]$. Given $c < c^*$, we fix $\bar{c} \in (c, c^*)$. For any $\epsilon \in \bar{\mathcal{C}}_\beta$ with $\epsilon \gg 0$, let the integer m, the large number B, and the sequence e_n be defined as in Lemmas 2.13 and 2.14. Since $Q^n[\sigma] \to \beta$ as $n \to \infty$, there is some l such that $\sigma_n = Q^n[\sigma] \gg Q_B^m[\alpha]$ for all $n \geq l$. Let the support of e_0 be contained in the interval with center at the origin and radius R_0. There is some r_σ such that if $u_0(\cdot, x) \geq \sigma(\cdot)$ for $|x| \leq r_\sigma$, then $u_l(\cdot, x) \geq Q_B^m[\alpha]$ for $|x| \leq R_0$. In particular, $u_l(\cdot, x) \geq e_0(\cdot, x)$. Since Q and Q_B are order preserving and $Q[v] > Q_B[v]$ for any $v \in \mathcal{C}_\beta$, Proposition 2.3 implies that $u_{l+n} \geq e_n$ for all $n \geq 0$. Thus, $u_{l+n}(\cdot, x) \geq Q_B^m[\alpha]$ if $|x| \leq (n + A)\bar{c} - 1 - m(B + \bar{c})$.

By Lemma 2.13, there is an integer $n_1 = n_1(\epsilon)$ such that $Q_B^{n_1+m}[\alpha] \gg \beta - \epsilon$. Since $c < \bar{c}$, there is some integer $N = N(c, n_1(\epsilon))$ such that for any $n \geq N$, if $|x_1| \leq (l + n + n_1)c$, then $|x_1| \leq (n + A)\bar{c} - 1 - m(B + \bar{c}) - n_1 B$. Thus, $|x_1 - x| \leq n_1 B$ implies $|x| \leq (n + A)\bar{c} - 1 - m(B + \bar{c})$. Therefore, for such x, we have $u_{l+n}(\cdot, x) \geq Q_B^m[\alpha]$, and hence

$$\begin{aligned} u_{l+n+n_1}(\theta, x_1) \geq Q_B^{n_1}[e_n](\theta, x_1) &= Q_B^{m+n_1}[\alpha](\theta) \\ &\gg \beta(\theta) - \epsilon(\theta) \quad \forall \theta \in [-\tau, 0]. \end{aligned}$$

Since $\bar{c} \in (c, c^*)$ is arbitrary, it follows that for any $\epsilon(\theta) \gg 0$, there is some $n_\epsilon = l + N + n_1$ such that for any $n \geq n_\epsilon$ and any x with $|x| \leq nc$, we have $u_n(\cdot, x) \gg \beta - \epsilon$. This completes the proof. □

We call c^* the *asymptotic speed of spread* (in short, *spreading speed*) of a discrete-time semiflow $\{Q^n\}_{n=0}^\infty$ on $\mathcal{C}_\beta$ provided that Theorems 2.11 and 2.15 hold. Moreover, a map $Q : \mathcal{C}_\beta \to \mathcal{C}_\beta$ is said to be *subhomogeneous* if $Q[\rho v] \geq \rho Q[v]$ for all $\rho \in [0, 1]$ and $v \in \mathcal{C}_\beta$.

COROLLARY 2.16 *Suppose that all assumptions of Theorem 2.15 hold. If, in addition, Q is subhomogeneous on $\mathcal{C}_\beta$, then we can choose r_σ in Theorem 2.15 to be independent of $\sigma \gg 0$.*

PROOF: Given $c < c^*$, we choose $c' \in (c, c^*)$. Fix $\sigma_0 \in \bar{\mathcal{C}}_\beta$ with $\sigma_0 \gg 0$. Thus, there exists $r_{\sigma_0} > 0$ such that if $u_0(\cdot, x) \geq \sigma_0$ for $x \in [-r_{\sigma_0}, r_{\sigma_0}]$, then $\lim_{n\to\infty, |x|\leq nc'} u_n(\theta, x) = \beta(\theta)$ uniformly for $\theta \in [-\tau, 0]$.

For any $v_0 \in \mathcal{C}_\beta$, if there is some $\sigma \in \bar{\mathcal{C}}_\beta$ with $\sigma \gg 0$ such that $v_0(\cdot, x) \geq \sigma$ for $x \in [-r_{\sigma_0}, r_{\sigma_0}]$, then there is some $\rho \in (0, 1]$ such that $v_0(\cdot, x) \geq \rho\sigma_0$ for $x \in [-r_{\sigma_0}, r_{\sigma_0}]$. Since $u_0(\cdot, x) = \frac{1}{\rho} v_0(\cdot, x) \geq \sigma_0$ for $x \in [-r_{\sigma_0}, r_{\sigma_0}]$, we have $\lim_{n\to\infty, |x|\leq nc'} u_n(\theta, x) = \beta(\theta)$ uniformly for $\theta \in [-\tau, 0]$.

Note that $v_n = Q^n[v_0] \geq \rho Q^n[u_0]$. This implies that for any $r > 0$, there is some n_0 such that $v_{n_0}(\cdot, x) \geq \rho\beta/2$ on $[-r, r]$. Moreover, there is some $r_{\rho\beta/2}$ such that if $u_0(\cdot, x) \geq \rho\beta/2$ for $x \in [-r_{\rho\beta/2}, r_{\rho\beta/2}]$, then $\lim_{n\to\infty, |x|\leq nc'} u_n(\theta, x) = \beta(\theta)$ uniformly for $\theta \in [-\tau, 0]$. Let $r = r_{\rho\beta/2}$. It then follows that

$$\lim_{\substack{n\to\infty \\ |x|\leq nc'}} v_{n+n_0}(\theta, x) = \beta(\theta)$$

uniformly for $\theta \in [-\tau, 0]$. Since for sufficiently large n, $(n + n_0)c < nc'$, we have $\lim_{n\to\infty, |x|\leq nc} v_n(\theta, x) = \beta(\theta)$ uniformly for $\theta \in [-\tau, 0]$. □

Finally, we extend our results on spreading speeds to continuous-time semiflows. Recall that a family of operators $\{Q_t\}_{t=0}^{\infty}$ is said to be a *semiflow* on $\mathcal{C}_\beta$ provided Q_t has the following properties:

(1) $Q_0(v) = v \; \forall v \in \mathcal{C}_\beta$.
(2) $Q_{t_1}[Q_{t_2}[v]] = Q_{t_1+t_2}[v] \;\; \forall t_1, t_2 \geq 0, v \in \mathcal{C}_\beta$.
(3) $Q(t, v) := Q_t(v)$ is continuous in (t, v) on $[0, \infty) \times \mathcal{C}_\beta$.

It is easy to see that property (3) holds if $Q(\cdot, v)$ is continuous on $[0, +\infty)$ for each $v \in \mathcal{C}_\beta$ and $Q(t, \cdot)$ is uniformly continuous for t in bounded intervals in the sense that for any $v_0 \in \mathcal{C}_\beta$, bounded interval I, and $\epsilon > 0$, there exists $\delta = \delta(v_0, I, \epsilon) > 0$ such that if $d(v, v_0) < \delta$, then $d(Q_t[v], Q_t[v_0]) < \epsilon$ for all $t \in I$.

THEOREM 2.17 *Let $\{Q_t\}_{t=0}^{\infty}$ be a semiflow on $\mathcal{C}_\beta$ with $Q_t[0] = 0$ and $Q_t[\beta] = \beta$ for all $t \geq 0$. Suppose that $Q = Q_1$ satisfies all hypotheses* (A1)–(A5)*, and Q_t satisfies* (A1) *for any $t > 0$. Let c^* be the asymptotic speed of spread of Q_1. Then the following statements are valid*:

(i) *For any $c > c^*$, if $v \in \mathcal{C}_\beta$ with $0 \leq v \ll \beta$ and $v(\cdot, x) = 0$ for x outside a bounded interval, then $\lim_{t\to\infty, |x|\geq tc} Q_t[v](\theta, x) = 0$ uniformly for $\theta \in [-\tau, 0]$.*
(ii) *For any $c < c^*$ and $\sigma \in \bar{\mathcal{C}}_\beta$ with $\sigma \gg 0$, there is a positive number r_σ such that if $v \in \mathcal{C}_\beta$ and $v(\cdot, x) \gg \sigma$ for x on an interval of length $2r_\sigma$, then $\lim_{t\to\infty, |x|\leq tc} Q_t[v](\theta, x) = \beta(\theta)$ uniformly for $\theta \in [-\tau, 0]$. If, in addition, Q_1 is subhomogeneous, then r_σ can be chosen to be independent of $\sigma \gg 0$.*

PROOF: First, it is easy to see that for any $v_n \to 0$, $Q_t[v_n] \to 0$ uniformly for $t \in [0, 1]$. In other words, for any $\epsilon > 0$ and any bounded interval I, there exists $\delta > 0$ and a sufficiently large positive number r such that if $v(\theta, x) < \delta$ for $x \in [-r, r]$ and $\theta \in [-\tau, 0]$, then $|Q_t[v](\theta, x)| < \epsilon$ for any $x \in I$, $\theta \in [-\tau, 0]$, and $t \in [0, 1]$. In particular, for any $\epsilon > 0$, we can find a sufficiently large positive number r such that for any $x_0 \in \mathbb{R}$, if $v(\theta, x) < \delta$ for $x \in [-r + x_0, r + x_0]$, $\theta \in [-\tau, 0]$, then $|Q_t[v](\theta, x_0)| < \epsilon$ for any $\theta \in [-\tau, 0]$, $t \in [0, 1]$.

For any $v \in \mathcal{C}_\beta$ with $0 \leq v \ll \beta$ and $v = 0$ outside a bounded subset of $[-\tau, 0] \times \mathbb{R}$ and any $c > c^*$, we have $\lim_{n\to\infty, |x|\geq nc} Q_n[v](\theta, x) = 0$ uniformly for $\theta \in [-\tau, 0]$. Hence, for the δ fixed above, we can find an integer N such that if $n \geq N$, then $|Q_n[v](\theta, x)| < \delta$ for any $\theta \in [-\tau, 0]$ and $|x| \geq nc$. Therefore, $|Q_t[v](\theta, x)| < \epsilon$ for any $n \geq N$, $t \in [n, n+1]$, $\theta \in [-\tau, 0]$, $|x| \geq nc + r$. For any $\rho > 0$, there is an integer N' such that if $n \geq N'$ and $t \in [n, n+1]$, then $t(c+\rho) > nc + r$. Thus, $|Q_t[v](\theta, x)| < \epsilon$ for any $t \geq \max(N, N')$ and $|x| \geq t(c+\rho)$. Since $c > c^*$ and $\rho > 0$ are arbitrary, conclusion (i) holds. Conclusion (ii) can be proved in a similar way. □

3 Estimates of Spreading Speeds

In this section, we discuss the spreading speeds for linear operators and then use them to estimate the spreading speeds for nonlinear maps and continuous semiflows.

Let $M : \mathcal{C} \to \mathcal{C}$ be a linear operator with the following properties:

(C1) M is continuous with respect to the compact open topology.

(C2) M is a positive operator; that is, $M[v] \geq 0$ whenever $v > 0$.

(C3) M satisfies (A3) with $\mathcal{C}_\beta$ replaced by any uniformly bounded subset of $\mathcal{C}$.

(C4) $M[\mathcal{R}[u]] = \mathcal{R}[M[u]]$, $T_y[M[u]] = M[T_y[u]]$ $\forall u \in \mathcal{C},\ y \in \mathcal{H}$.

(C5) M can be extended to a linear operator on the linear space $\tilde{\mathcal{C}}$ of all functions $v \in C([-\tau, 0] \times \mathcal{H}, \mathbb{R}^k)$ having the form

$$v(\theta, x) = v_1(\theta, x)e^{\mu_1 x} + v_2(\theta, x)e^{\mu_2 x}, \quad v_1, v_2 \in \mathcal{C},\ \mu_1, \mu_2 \in \mathbb{R},$$

such that if $v_n, v \in \tilde{\mathcal{C}}$ and $v_n(\theta, x) \to v(\theta, x)$ uniformly on any bounded set, then $M[v_n](\theta, x) \to M[v](\theta, x)$ uniformly on any bounded set.

As we remarked on (A1) in Section 2, hypothesis (C4) implies that $M[v] \in \bar{\mathcal{C}}$ whenever $v \in \bar{\mathcal{C}}$, and hence M is also a linear operator on $\bar{\mathcal{C}}$.

Define the linear map $B_\mu : \bar{\mathcal{C}} \to \bar{\mathcal{C}}$ by

$$B_\mu[\alpha](\theta) = M[\alpha e^{-\mu x}](\theta, 0) \quad \forall \theta \in [-\tau, 0].$$

In particular, $B_0 = M$ on $\bar{\mathcal{C}}$. If $\alpha_n, \alpha \in \bar{\mathcal{C}}$ and $\alpha_n \to \alpha$ as $n \to \infty$, then $\alpha_n(\theta)e^{-\mu x} \to \alpha(\theta)e^{-\mu x}$ uniformly on any bounded subset of $[-\tau, 0] \times \mathcal{H}$. Thus, $B_\mu[\alpha_n] = M[\alpha_n e^{-\mu x}](\cdot, 0) \to M[\alpha e^{-\mu x}](\cdot, 0) = B_\mu[\alpha]$, and hence B_μ is continuous. Moreover, B_μ is a positive operator on $\bar{\mathcal{C}}$.

In this section, we assume that

(C6) For any $\mu \geq 0$, B_μ is a positive operator, and there is n_0 such that

$$B_\mu^{n_0} = \underbrace{B_\mu \circ \cdots \circ B_\mu}_{n_0}$$

is a compact and strongly positive linear operator on $\bar{\mathcal{C}}$.

LEMMA 3.1 *Let B be a bounded and positive linear operator on the ordered Banach space (X, P) with the positive cone P having nonempty interior* Int(P). *If there is a positive integer n such that B^n is compact and strongly positive on X* (*i.e.,*

$B^n(P \setminus \{0\}) \subset \mathrm{Int}(P)$), *then the spectral radius* λ *of* B *is a simple eigenvalue of* B *having a strongly positive eigenvector, and the modulus of any other eigenvalue is less than* λ.

PROOF: Let r be the spectral radius of B^n. By the classical Krein-Rutman theorem, it follows that $r > 0$, and r is the unique eigenvalue of B^n having a positive eigenvector. Moreover, r is a simple eigenvalue of B^n. Let $v \gg 0$ be an eigenvector of B^n associated with r. From the positivity of B and the property of r, it is easy to see that $v' := B[v] > 0$. Then $0 \ll B^n[v'] = B^n[B[v]] = B[B^n[v]] = rv'$, and hence v' is a strongly positive eigenvector of B^n associated with r. Thus, $B[v] = v' = \lambda v$ for some $\lambda > 0$, which implies that λ is a positive eigenvalue of B with eigenvector $v \gg 0$. Since $B^n[v] = \lambda^n v$, it follows from the aforementioned property of r that $\lambda^n = r$, and hence $\lambda = r^{1/n}$ is the spectral radius of B. Given an eigenvalue μ of B, let $\hat{v} \neq 0$ be an eigenvector associated with μ. Then $B[\hat{v}] = \mu\hat{v}$, and hence $B^n[\hat{v}] = \mu^n\hat{v}$. Consequently, either $|\mu| < r^{1/n}$, or $\mu = r^{1/n}$ and $\hat{v}$ is a multiple of v. This completes the proof. □

Let $\lambda(\mu)$ be the principal eigenvalue of B_μ and $\zeta_\mu(\cdot) = \zeta(\mu, \cdot)$ be a strongly positive eigenfunction associated with $\lambda(\mu)$.

LEMMA 3.2 *For any integer* n, B_μ *and* $\lambda(\mu)$ *are* n *times differentiable in* μ. *Moreover, we can choose appropriate* ζ_μ *such that* ζ_μ *is also* n *times differentiable in* μ.

PROOF: For any $\alpha \in \bar{\mathcal{C}}$ with $\|\alpha\| = 1$. Fix $\mu_0 \geq 0$, and for any $\mu > \mu_0$, set

$$h(\mu, x) = \frac{e^{-\mu x} - e^{-\mu_0 x}}{\mu - \mu_0} - xe^{-\mu_0 x}.$$

For any x, there is some $\mu' \in (\mu_0, \mu)$ such that $(e^{-\mu x} - e^{-\mu_0 x})/(\mu - \mu_0) = xe^{-\mu' x}$ and hence

$$h(\mu, x) = xe^{-\mu' x} - xe^{-\mu_0 x}.$$

Thus, $h(\mu, x) \leq 0$ for $x \geq 0$, and $h(\mu, x) \geq 0$ for $x \leq 0$. Define

$$h^+(\mu, x) = \begin{cases} 0, & x \geq 0, \\ h(\mu, x), & x \leq 0, \end{cases}$$

and $h^-(\mu, x) = h(\mu, x) - h^+(\mu, x)$. Then $h^+(\mu, x)$ and $h^-(\mu, x) \to 0$ as $\mu \to \mu_0$ uniformly for x in any bounded subset of $\mathbb{R}$. Define the linear operator L on $\bar{\mathcal{C}}$ by

$$L[\alpha](\theta) = M[\alpha x e^{\mu_0 x}](\theta, 0) \quad \forall \theta \in [-\tau, 0].$$

Then L is a continuous operator. For any $\alpha \in \bar{\mathcal{C}}$ with $-1 \leq \alpha \leq 1$, we have

$$\frac{B_\mu[\alpha] - B_{\mu_0}[\alpha]}{\mu - \mu_0} - L[\alpha]$$

$$= M[\alpha h(\mu, x)](\,\cdot\,, 0)$$

$$= M[\alpha h^+(\mu, x)](\cdot\,, 0) + M[\alpha h^-(\mu, x)](\cdot\,, 0)$$

$$\le M[h^+(\mu, x)](\cdot\,, 0) - M[h^-(\mu, x)](\cdot\,, 0) \to 0 \quad \text{as } \mu \to \mu_0.$$

Similarly,

$$\frac{B_\mu[\alpha] - B_{\mu_0}[\alpha]}{\mu - \mu_0} - L[\alpha] \ge -M[h^+(\mu, x)](\cdot\,, 0) + M[h^-(\mu, x)](\cdot\,, 0) \to 0 \quad \text{as } \mu \to \mu_0.$$

This implies

$$\lim_{\mu \to \mu_0} \frac{B_\mu[\alpha] - B_{\mu_0}[\alpha]}{\mu - \mu_0} = L[\alpha]$$

uniformly for all α with $|\alpha| \le 1$. For $\mu < \mu_0$, we have the same conclusion. Hence B_μ is differentiable in μ. By a similar argument, we can prove that B_μ is n times differentiable in μ for any n.

By Lemma 3.1, it follows that the spectral radius $\lambda(\mu)$ of B_μ is a simple eigenvalue of B_μ and the modulus of any other eigenvalue is less than $\lambda(\mu)$. Thus, the other two conclusions follow from the results in [17, sec. 7.1]. □

Note that the principal eigenvalue λ of B_μ can be characterized as

$$\lambda = \min_{\xi \gg 0} \max_{i,\theta} \frac{(B_\mu[\xi])_i(\theta)}{\xi_i(\theta)} = \max_{\xi \gg 0} \min_{i,\theta} \frac{(B_\mu[\xi])_i(\theta)}{\xi_i(\theta)}. \tag{3.1}$$

Define $\bar{M} : \mathcal{C}_{\zeta_0} \to \mathcal{C}_{\zeta_0}$ by

$$\bar{M}[u] = \min\{\zeta_0, M[u]\}.$$

In what follows, we prove that $\bar{M}$ has the asymptotic speed of spread $\bar{c}^*$ provided the following condition is satisfied:

(C7) The principal eigenvalue $\lambda(0)$ of B_0 is larger than 1.

LEMMA 3.3 *$\bar{M} : \bar{\mathcal{C}}_{\zeta_0} \to \bar{\mathcal{C}}_{\zeta_0}$ admits exactly two fixed points* 0 *and* ζ_0.

PROOF: It is obvious that $\bar{M}$ maps $\bar{\mathcal{C}}_{\zeta_0}$ into $\bar{\mathcal{C}}_{\zeta_0}$. Assume, for the sake of contradiction, that $\bar{M}$ has a fixed point $v \in \bar{\mathcal{C}}_{\zeta_0}$ such that $0 < v < \zeta_0$. Since $\zeta_0 \gg 0$, we can choose a real number $\rho \in (0, 1)$ such that $M^n[\rho v] \le \zeta_0$ for all $0 \le n \le n_0$. It then follows that $v = \bar{M}^{n_0}[v] \ge \bar{M}^{n_0}[\rho v] = M^{n_0}[\rho v] = B_0^{n_0}[\rho v] \gg 0$, and hence $v \ge \rho' \zeta_0$ for some real number $\rho' \in (0, 1)$. Define $m := \inf\{n \ge 1 : (\lambda(0))^n \rho' \ge 1\}$. Clearly, condition (C7) implies that m is a finite positive integer. Note that $M^n[\rho'\zeta_0] = (\lambda(0))^n \rho' \zeta_0$ for all $n \ge 0$. Since $M^n[\rho'\zeta_0] < \zeta_0$ for all $0 \le n \le m - 1$, we have $\bar{M}^m[\rho'\zeta_0] = \min\{\zeta_0, M^m[\rho'\zeta_0]\} = \zeta_0$. Thus, we obtain $v = \bar{M}^m[v] = \zeta_0$, a contradiction. □

It is easy to see that $\bar{M}$ satisfies (A1), (A2), (A4), and (A5) with $\beta = \zeta_0$. We can also define the operator R_c and the sequence a_n with Q replaced by $\bar{M}$. In this case, we have

$$\begin{aligned} R_c[a](\theta, s) &= \max\{\phi(\theta, s), T_{-c}[\bar{M}[a]](\theta, s)\} \\ &= \max\{\phi(\theta, s), \min\{\zeta_0, T_{-c}[M[a]](\theta, s)\}\}. \end{aligned} \tag{3.2}$$

As argued in the proof of Lemma 2.5, we can show that $\{a_n(c; \cdot, s) : n \geq 0,\ c, s \in \mathbb{R}\}$ is a family of equicontinuous functions by using (3.2) and hypothesis (C3) for M. Moreover, all the conclusions for Q can be proved for $\bar{M}$ by a similar argument. By Theorems 2.11 and 2.15, it then follows that $\bar{M}$ has a spreading speed $\bar{c}^*$. It is also easy to see that $\bar{c}^*$ is independent of the choice of the eigenfunction associated with $\lambda(0)$. For convenience, we call $\bar{c}^*$ the spreading speed of M.

For any $\rho \in [0, 1]$, we have

$$\begin{aligned} \bar{M}[\rho v] &= \min\{\zeta_0, M[\rho v]\} = \min\{\zeta_0, \rho M[v]\} \\ &\geq \min\{\rho\zeta_0, \rho M[v]\} = \rho \min\{\zeta_0, M[v]\} = \rho \bar{M}[v]; \end{aligned}$$

that is, $\bar{M}$ is subhomogeneous. By Corollary 2.16, we have the following result:

COROLLARY 3.4 *For any $c < \bar{c}^*$, there exists $r > 0$ such that if there is some $\sigma \in \bar{\mathcal{C}}_\beta$ with $\sigma \gg 0$ and $u_0(\cdot, x) \geq \sigma$ for x on an interval of length $2r$, then*

$$\lim_{\substack{n\to\infty \\ |x|\leq nc}} \bar{M}^n[u_0](\theta, x) = \zeta_0(\theta)$$

uniformly for $\theta \in [-\tau, 0]$.

THEOREM 3.5 *Let c^* be the spreading speed of Q. Assume that there is a sequence of linear operators M_n satisfying* (C1)–(C7) *such that the spreading speed c_n^* of M_n converges to c^* as $n \to \infty$ and that for each n there is $\sigma_n \in \bar{\mathcal{C}}_\beta$ with $\sigma_n > 0$ such that $M_n[v] \leq Q[v]$ for any $v \in \mathcal{C}_\beta$ with $v \leq \sigma_n$. Then we can choose r_σ in Theorem* 2.15 *to be independent of $\sigma \gg 0$.*

PROOF: Since $c_n^* \to c^*$ as $n \to \infty$, there is an integer m such that $c_m^* > c$. Choose a principal eigenvector ζ_0 of M_m in $\bar{\mathcal{C}}$ such that $0 \ll \zeta_0 \leq \sigma_m$, and let $M := M_m$. We claim that for any $v \in \mathcal{C}_\beta$ with $v \leq \zeta_0$ and $Q^n[v] \geq \bar{M}^n[v]$ $\forall n \geq 1$. Indeed, $Q[v] \geq \bar{M}[v]$. Assume that $Q^n[v] \geq \bar{M}^n[v]$ for some n. Then

$$\begin{aligned} Q^{n+1}[v] &= Q[Q^n[v]] \\ &\geq Q[\min\{\zeta_0, Q^n[v]\}] \geq \bar{M}[\min\{\zeta_0, Q^n[v]\}] \\ &\geq \bar{M}[\min\{\zeta_0, \bar{M}^n[v]\}] = \bar{M}[\bar{M}^n[v]] = \bar{M}^{n+1}[v]. \end{aligned}$$

By induction, our claim holds for all $n \geq 1$.

By Theorem 2.15, it follows that there is some $r_{\zeta_0/2}$ such that $v(\theta, x) \geq \zeta_0(\theta)/2$ for $\theta \in [-\tau, 0]$, $x \in [-r_{\zeta_0/2}, r_{\zeta_0/2}]$, and hence

$$\lim_{\substack{n\to\infty \\ |x|\leq nc}} Q^n[v](\theta, x) = \beta(\theta)$$

uniformly for $\theta \in [-\tau, 0]$. By Corollary 3.4, there is some $r > 0$ such that if there is some $\sigma \in \bar{\mathcal{C}}_\beta$ with $\sigma \gg 0$ and $u_0(\cdot, x) \geq \sigma$ for $x \in [-r, r]$, then there is some n_0 such that

$$Q^{n_0}[u_0](\theta, x) \geq \bar{M}^{n_0}[u_0](\theta, x) \geq \frac{\zeta_0(\theta)}{2} \quad \forall \theta \in [-\tau, 0],\ x \in [-r_{\zeta_0/2}, r_{\zeta_0/2}].$$

Thus, we have $\lim_{n\to\infty, |x|\leq nc} Q^{n+n_0}[u_0](\theta, x) = \beta(\theta)$ uniformly for $\theta \in [-\tau, 0]$.

For any $c' < c$, when n is sufficiently large, we have $(n + n_0)c' < nc$, and hence $\lim_{n\to\infty, |x|\leq nc'} u_n(\theta, x) = \beta(\theta)$ uniformly for $\theta \in [-\tau, 0]$. Since $c, c' < c^*$ are arbitrary, our theorem holds. □

The following remark is a simple application of Theorem 3.5 to continuous-time semiflows.

Remark 3.6. Let the assumptions of Theorem 2.17 hold. If, in addition, Q_1 satisfies the conditions of Theorem 3.5, then r_σ in Theorem 2.17(ii) can be chosen to be independent of $\sigma \gg 0$.

LEMMA 3.7 *$\lambda(\mu)$ is log convex on $\mathbb{R}$.*

PROOF: We use an argument similar to that in [21]. By the Riesz representation theorem, it follows that for any $\theta_0 \in [-\tau, 0]$, there exist bounded symmetric nonnegative measures $m_{ij}^{\theta_0}$ on $[-\tau, 0] \times \mathbb{R}$ such that for any $v \in \mathcal{C}$,

$$M_i[v](\theta_0, x) = \sum_{j=1}^{k} \int_{-\infty}^{\infty} v_j(\theta, x - y) m_{ij}^{\theta_0}(d\theta\, dy).$$

By the definition of $\lambda(\mu)$, we see that there exist positive eigenvectors $\nu, \eta \in \bar{\mathcal{C}}$ such that

$$\lambda(\mu_1) = \frac{1}{\nu_i(\theta_0)} \left(\sum_{j=1}^{k} \int_{-\infty}^{\infty} \nu_j(\theta) e^{\mu_1 y} m_{ij}^{\theta_0}(d\theta\, dy) \right),$$

$$\lambda(\mu_2) = \frac{1}{\eta_i(\theta_0)} \left(\sum_{j=1}^{k} \int_{-\infty}^{\infty} \eta_j(\theta) e^{\mu_2 y} m_{ij}^{\theta_0}(d\theta\, dy) \right),$$

for any $\theta_0 \in [-\tau, 0]$, $1 \leq i \leq k$. From the Hölder inequality, it follows that for $0 < t < 1$ and each $\theta_0 \in [-\tau, 0]$, $1 \leq i \leq k$, we have

$$\begin{aligned} &\lambda(\mu_1)^t \lambda(\mu_2)^{1-t} \\ &\quad \geq \sum_{j=1}^{k} \int_{-\infty}^{\infty} \left(\frac{\nu_j(\theta)}{\nu_i(\theta_0)} \right)^t e^{\mu_1 y t} \left(\frac{\eta_j(\theta)}{\eta_i(\theta_0)} \right)^{1-t} e^{\mu_2 y(1-t)} m_{ij}^{\theta_0}(d\theta\, dy) \\ &\quad = \sum_{j=1}^{k} \int_{-\infty}^{\infty} \left(\frac{\nu_j(\theta)}{\nu_i(\theta_0)} \right)^t \left(\frac{\eta_j(\theta)}{\eta_i(\theta_0)} \right)^{1-t} e^{\mu_1 y t + \mu_2 y(1-t)} m_{ij}^{\theta_0}(d\theta\, dy). \end{aligned}$$

Let $\xi^i(\theta) = \nu^i(\theta)^t \eta^i(\theta)^{1-t}$. Then

$$\lambda(\mu_1)^t \lambda(\mu_2)^{1-t} \geq \frac{(B_{t\mu_1+(1-t)\mu_2}[\xi])_i(\theta_0)}{\xi_i(\theta_0)}$$

for all $1 \leq i \leq k$, $\theta_0 \in [-\tau, 0]$. Now (3.1) completes the proof. □

Let $\Phi(\mu) = \frac{1}{\mu} \ln \lambda(\mu)$ and $\Psi(\mu) = \lambda'(\mu)/\lambda(\mu)$. It then follows that

$$\Psi' \geq 0, \qquad (\mu\Phi(\mu))' = \Psi(\mu),$$

$$\Phi'(\mu) = \frac{1}{\mu}[\Psi(\mu) - \Phi(\mu)], \qquad (\mu^2\Phi')' = \mu\Psi'(\mu) \geq 0.$$

The proof of the subsequent lemma is straightforward.

LEMMA 3.8 *The following statements are valid*:

(i) $\Phi(\mu) \to \infty$ *as* $\mu \downarrow 0$.
(ii) $\Phi(\mu)$ *is decreasing near* 0.
(iii) $\Phi'(\mu)$ *changes sign at most once on* $(0, \infty)$.
(iv) Ψ *is increasing and* $\lim_{\mu\to\infty} \Phi(\mu) = \lim_{\mu\to\infty} \Psi(\mu)$, *where the limits may be infinite.*

We say that M has *compact support* provided there is some ρ such that for any $\alpha \in \mathcal{C}$, $M[\alpha](\theta, x)$ depends only on the value of α in $[-\tau, 0] \times [x - \rho, \rho + x]$.

PROPOSITION 3.9 *Let* $\bar{c}^*$ *be the asymptotic speed of spread of* $\bar{M}$. *Then* $\bar{c}^* \leq \inf_{\mu>0} \Phi(\mu)$. *If, in addition, either* M *has compact support, or the infimum of* $\Phi(\mu)$ *is attained at some finite value* μ^* *and* $\Phi(+\infty) > \Phi(\mu^*)$, *then* $\bar{c}^* = \inf_{\mu>0} \Phi(\mu)$.

PROOF: For each $\mu > 0$, define $w = \min\{\zeta_0, \bar{w}\}$, where $\bar{w} = \zeta_\mu e^{-\mu s}$. Then

$$\begin{aligned} \bar{M}[T_{-\Phi(\mu)}[w]](\theta, s) &= \bar{M}[T_{-\Phi(\mu)-s}[w]](\theta, 0) \\ &\leq M[T_{-\Phi(\mu)-s}[w]](\theta, 0) \\ &\leq M[T_{-\Phi(\mu)-s}[\bar{w}]](\theta, 0) = \zeta_\mu e^{-\mu s}, \end{aligned}$$

and hence $\bar{M}[T_{-\Phi(\mu)}[w]] \leq w$. Let α be chosen as in (A5) with β replaced by ζ_0, and fix ϕ such that (B1)–(B3) hold. Moreover, we can define R_c with $Q = \bar{M}$. Thus, we have $R_c[w] \leq w$ for any $c \geq \Phi(\mu)$. Since $a_0 = \phi \leq w$, the monotonicity of R_c implies that $a_n \leq w$ for $n \geq 1$. Letting $n \to \infty$, we then have $a \leq w$, and hence $a(c; \cdot, \infty) = 0$ if $c \geq \Phi(\mu)$. Thus, $\bar{c}^* \leq \inf_{\mu>0} \Phi(\mu)$.

Next we show that $\bar{c}^* \geq \inf_{\mu>0} \Phi(\mu)$. First, consider the case where M has compact support. Fix $\mu \in (0, \mu^*)$ where the infimum of $\Phi(\mu)$ is attained at μ^*, and let

$$\kappa_\mu^i(\theta) = \kappa^i(\mu, \theta) := \frac{\partial \zeta^i(\theta, \mu)}{\partial \mu} \frac{1}{\zeta^i(\theta, \mu)}.$$

For convenience, we write $\kappa^i(\theta) = \kappa^i_\mu(\theta)$. Define $v = (v^1, \dots, v^k)$ with

$$v^i(\theta, s) = \begin{cases} \epsilon \zeta^i_\mu(\theta) e^{-\mu s} \sin r(s - \kappa^i(\theta)), & 0 \le s - \kappa^i(\theta) \le \frac{\pi}{r}, \\ 0 & \text{otherwise,} \end{cases}$$

where ϵ and r are sufficiently small positive numbers. Let $\xi = (\xi^1, \dots, \xi^k)$ with $\xi^i(\theta, s) = \zeta^i_\mu(\theta) e^{-\mu s} \sin r(-s + \kappa^i(\theta))$, and $\omega = (\omega^1, \dots, \omega^k)$ with $\omega^i(\theta, s) = \zeta^i_\mu(\theta) e^{-\mu s} \cos r(-s + \kappa^i(\theta))$. Then $\omega(\theta, s)$ converges to $\zeta_\mu e^{-\mu s}$ uniformly on any compact subset of $[-\tau, 0] \times \mathbb{R}$ as $r \to 0$.

Define z^i by

$$z^i(r, \theta) = \frac{1}{r} \tan^{-1} \frac{(M[\xi])_i(\theta, 0)}{(M[\omega])_i(\theta, 0)}.$$

Then z^i is a family of equicontinuous and uniformly bounded functions of θ if r is regarded as a parameter in (0, 1]. Moreover, we have

$$\begin{aligned} \lim_{r\downarrow 0} z^i(r, \theta_0) &= \lim_{r\downarrow 0} \frac{(M[\xi])_i(\theta_0, 0)}{r(M[\omega])_i(\theta_0, 0)} \\ &= \lim_{r\downarrow 0} \frac{\sum_{j=1}^k \int_{-\infty}^{\infty} \zeta^j_\mu(\theta) e^{\mu y} \sin r(y + \kappa^j(\theta)) m^{\theta_0}_{ij}(d\theta\, dy)}{r \sum_{j=1}^k \int_{-\infty}^{\infty} \zeta^j_\mu(\theta) e^{\mu y} \cos r(y + \kappa^j(\theta)) m^{\theta_0}_{ij}(d\theta\, dy)} \\ &= \lim_{r\downarrow 0} \frac{\sum_{j=1}^k \int_{-\infty}^{\infty} \zeta^j_\mu(\theta) e^{\mu y} \cdot \frac{\sin r(y+\kappa^j(\theta))}{r} m^{\theta_0}_{ij}(d\theta\, dy)}{\sum_{j=1}^k \int_{-\infty}^{\infty} \zeta^j_\mu(\theta) e^{\mu y} \cos r(y + \kappa^j(\theta)) m^{\theta_0}_{ij}(d\theta\, dy)}. \end{aligned}$$

By the Lebesgue dominated convergence theorem,

$$\begin{aligned} \lim_{r\downarrow 0} z^i(r, \theta_0) &= \frac{\sum_{j=1}^k \int_{-\infty}^{\infty} \zeta^j_\mu(\theta) e^{\mu y} (y + \kappa^j(\theta)) m^{\theta_0}_{ij}(d\theta\, dy)}{\sum_{j=1}^k \int_{-\infty}^{\infty} \zeta^j_\mu(\theta) e^{\mu y} m^{\theta_0}_{ij}(d\theta\, dy)} \\ &= \frac{\sum_{j=1}^k \int_{-\infty}^{\infty} \frac{\partial(\zeta^j_\mu(\theta) e^{\mu y})}{\partial \mu} m^{\theta_0}_{ij}(d\theta\, dy)}{\lambda(\mu) \zeta^i_\mu(\theta_0)} \end{aligned}$$

$$= \frac{\partial(\sum_{j=1}^{k} \int_{-\infty}^{\infty} \zeta_\mu^j(\theta) e^{\mu y} m_{ij}^{\theta_0}(d\theta\, dy))/\partial\mu}{\lambda(\mu)\zeta_\mu^i(\theta_0)}$$

$$= \frac{\frac{\partial(\lambda(\mu)\zeta_\mu^i(\theta_0))}{\partial\mu}}{\lambda(\mu)\zeta_\mu^i(\theta_0)} = \Psi(\mu) + \kappa^i(\theta_0), \quad 1 \le i \le k,$$

uniformly for $\theta_0 \in [-\tau, 0]$.

Choose r so small that $r(\rho + |z^i(r, \theta_0)| + |\kappa^j(\theta)|) < \pi$ for all $1 \le i, j \le k$, $\theta_0, \theta \in [-\tau, 0]$. If $0 \le s - \kappa^i(\theta_0) \le \frac{\pi}{r}$ and $-\rho < x < \rho$, then

$$-\frac{\pi}{r} \le x + s - \kappa^i(\theta_0) + z^i(r, \theta_0) - \kappa^j(\theta) \le \frac{2\pi}{r}.$$

Therefore

$$\begin{aligned} &v^j(\theta, x + s - \kappa^i(\theta_0) + z^i(r, \theta_0)) \\ &\quad \ge \epsilon \zeta_\mu^j(\theta) e^{-\mu(x+s-\kappa^i(\theta_0)+z^i(r,\theta_0))} \cdot \sin r(x + s - \kappa^i(\theta_0) + z^i(r, \theta_0) - \kappa^j(\theta)) \end{aligned}$$

for $1 \le j \le k$. Let $u = (u^1, \dots, u^k)$ with

$$u^i(\theta, s) = \epsilon \zeta_\mu^i(\theta) e^{-\mu s} \sin r(s - \kappa^i(\theta)).$$

Thus, we have

$$(M[T_{\kappa^i(\theta_0)-z^i(r,\theta_0)}[v]])_i(\theta_0, s) \ge (M[T_{\kappa^i(\theta_0)-z^i(r,\theta_0)}[u]])_i(\theta_0, s)$$

and hence

$$(\bar{M}[T_{\kappa^i(\theta_0)-z^i(r,\theta_0)}[v]])_i(\theta_0, s) \ge \min\{\zeta_0^i(\theta_0), (M[T_{\kappa^i(\theta_0)-z^i(r,\theta_0)}[u]])_i(\theta_0, s)\}.$$

Moreover,

$$\begin{aligned} &(M[T_{\kappa^i(\theta_0)-z^i(r,\theta_0)}[u]])_i(\theta_0, s) \\ &\quad = \epsilon e^{-\mu(s-\kappa^i(\theta_0)+z^i(r,\theta_0))} \big\{ M[\omega](\theta_0, 0) \sin r(s - \kappa^i(\theta_0) + z^i(r, \theta_0)) \\ &\qquad\qquad - M[\xi](\theta_0, 0) \cos r(s - \kappa^i(\theta_0) + z^i(r, \theta_0)) \big\} \\ &\quad = \epsilon e^{-\mu(s-\kappa^i(\theta_0)+z^i(r,\theta_0))} \sin r(s - \kappa^i(\theta_0))(\sec r z^i(r, \theta_0)) M[\omega](\theta_0, 0). \end{aligned}$$

Since $e^{-\mu z^i(r,\theta_0)}(\sec r z^i(r, \theta_0)) M[\omega](\theta_0, 0)$ converges to

$$e^{\mu[\Phi(\mu)-\Psi(\mu)]} e^{-\mu\kappa^i(\theta_0)} \zeta_\mu^i(\theta_0, 0) > \zeta_\mu^i(\theta_0, 0) e^{-\mu\kappa^i(\theta_0)}$$

as $r \downarrow 0$ uniformly for $\theta_0 \in [-\tau, 0]$, we have

$$(\bar{M}[T_{\kappa^i(\theta_0)-z^i(r,\theta_0)}[v]])_i(\theta_0, s) \ge v^i(\theta_0, s)$$

if r and ϵ are sufficiently small.

Let $\kappa := \max_{1 \le i \le k, \theta \in [-\tau, 0]} \kappa^i(\theta)$ and define

$$\varphi^i(\theta, s) = \begin{cases} v^i(\theta, \bar{s}^i(\theta)), & s \le \bar{s}^i(\theta) - \frac{\pi}{r} - \kappa, \\ v^i(\theta, s + \frac{\pi}{r} + \kappa), & s \ge \bar{s}^i(\theta) - \frac{\pi}{r} - \kappa, \end{cases}$$

where $\bar{s}^i(\theta)$ is the maximum point of $v^i(\theta, \cdot)$ on $\mathbb{R}$. Then φ is continuous and nonincreasing in s, and vanishes for $s \geq 0$. It is easy to see that

$$M[\varphi(\cdot, -\infty)] \geq \varphi(\cdot, -\infty)$$

and that φ satisfies (B1)–(B3) with $Q = \bar{M}$ and $\beta = \zeta_0$. Moreover, φ also has the property that $\varphi^i(\theta, s) = \max\{v^i(\theta, s-t) : t \leq -\frac{\pi}{r} - \kappa\}$. This implies that

$$\bar{M}[T_{\kappa^i(\theta_0)-z^i(r,\theta_0)}[\varphi]])_i(\theta_0, s) \geq \bar{M}[T_{\kappa^i(\theta_0)-z^i(r,\theta_0+t)}[v]])_i(\theta_0, s) \geq v^i(\theta_0, s-t)$$

for $t \leq -\frac{\pi}{r} - \kappa$. Therefore, we have

$$\bar{M}[T_{\kappa^i(\theta_0)-z^i(r,\theta_0)}[\varphi]])_i(\theta_0, s) \geq \varphi^i(\theta, s) \quad \forall s \in \mathbb{R}$$

for $1 \leq i \leq k$. Let $\bar{z}(r) = \min_{\theta,i}(-\kappa^i(\theta) + z^i(r,\theta))$. Then $\lim_{r\downarrow 0} \bar{z}(r) = \Psi(\mu)$ and $\bar{M}[T_{-\bar{z}}[\varphi]]) \geq \varphi$. It is easy to show that $\bar{z}(r) \leq \bar{c}^*$ for sufficiently small r, and hence $\Psi(\mu) \leq \bar{c}^*$ for $0 < \mu < \mu^*$. Thus, we have $\inf_{\mu>0} \Phi(\mu) = \Psi(\mu^*) \leq \bar{c}^*$.

In the case where M has no compact support, we define

$$M_l[u](\theta, y) = Q\left[T_{-y}[u] \cdot \varpi\left(\frac{|x|}{l}\right)\right](\theta, 0)$$

and

$$B^l_\mu[\alpha](\theta) = M_l[\alpha e^{-\mu x}](\theta, 0) = M\left[\alpha e^{-\mu x} \varpi\left(\frac{|x|}{l}\right)\right](\theta, 0).$$

We claim that $B^l_\mu \to B_\mu$ as $l \to \infty$. In fact, it is obvious that $B_\mu - B^l_\mu$ is a positive operator. For any α with $\|\alpha\| = 1$, since $-1 \leq \alpha(\theta) \leq 1$, we have

$$-(B_\mu - B^l_\mu)[1](\theta) \leq (B_\mu - B^l_\mu)[\alpha](\theta) \leq (B_\mu - B^l_\mu)[1](\theta) \quad \forall \theta \in [-\tau, 0].$$

This implies that $\|(B_\mu - B^l_\mu)[\alpha]\| \leq \|(B_\mu - B^l_\mu)[1]\|$, and hence $\|B_\mu - B^l_\mu\| = \|(B_\mu - B^l_\mu)[1]\|$. From the definition of $M[e^{-\mu x}]$, we obtain that $\|B_\mu - B^l_\mu\| = \|(B_\mu - B^l_\mu)[1]\| \to 0$ as $l \to \infty$.

Let $\lambda_l(\mu)$ be the principal eigenvalue of B^l_μ, and $\Phi_l(\mu) = \ln \lambda_l(\mu)/\mu$. Then $\lambda_l(\mu) \to \lambda(\mu)$, the principal eigenvalue of B_μ, uniformly for μ in any compact subset of $(0, +\infty)$ and $\Phi_l(\mu) \to \Phi(\mu)$ as $l \to \infty$. Since Φ achieves its minimum at some finite value μ^* and $\Phi(\infty) > \Phi(\mu^*)$, Φ_l also achieves its minimum at some finite value μ^*_l and $\mu^*_l \to \mu^*$. Thus, $\lim_{l\to\infty} \inf \Phi_l(\mu) = \inf \Phi(\mu)$. Note that M_l has compact support. By what we have proved, it follows that $\bar{c}^* \geq \inf_{\mu>0} \Phi(\mu)$. □

THEOREM 3.10 *Let Q be an operator on $\mathcal{C}_\beta$ satisfying* (A1)–(A5) *and c^* be defined as in Section 2. Assume that the linear operator M satisfies all hypotheses in Proposition* 3.9. *Then the following statements are valid*:

(i) *If $Q[u] \leq M[u]$ for all $u \in \mathcal{C}_\beta$, then $c^* \leq \inf_{\mu>0} \Phi(\mu)$.*
(ii) *If there is some $\eta \in \bar{\mathcal{C}}$ with $\eta \gg 0$ such that $Q[u] \geq M[u]$ for any $u \in \mathcal{C}_\eta$, then $c^* \geq \inf_{\mu>0} \Phi(\mu)$.*

PROOF: To prove the first statement, we choose the principal eigenvector ζ_0 of B_0 such that $\zeta_0 \gg \beta$. Let $\bar{c}^*$ be the spreading speed of $\bar{M}$. By Lemma 2.9 and Proposition 3.9, it follows that $c^* \leq \bar{c}^* = \inf_{\mu>0} \Phi(\mu)$. The second statement can be proved by choosing $\zeta_0 \ll \beta$. □

4 Traveling Waves

In this section, we show that the spreading speeds for monotone discrete and continuous-time semiflows coincide with the minimal wave speeds of their monotone traveling waves under appropriate assumptions.

For any real number c, we define the set

$$\mathcal{D}_c := \{x - mc : x \in \mathcal{H}, m \in \mathbb{N}\}.$$

We say that $W(\theta, x - nc)$ is a *traveling wave* of the map Q with the wave speed c if $W : [-\tau, 0] \times \mathcal{D}_c \to \mathbb{R}^k$ and $Q^n[W](\theta, x) = W(\theta, x - nc)$. We say that $W(\theta, x - nc)$ *connects* β *to* 0 if $W(\cdot, -\infty) = \beta$ and $W(\cdot, \infty) = 0$.

THEOREM 4.1 *Let Q satisfy* (A1)–(A5)*, and c^* be its asymptotic speed of spread. Then for any $c < c^*$, Q has no traveling wave $W(\theta, x - nc)$ connecting β to* 0*.*

PROOF: By Theorem 2.15, it follows that there is $r = r_{\beta/2}$ such that for any $u \in \mathcal{C}_\beta$ and $x_0 \in \mathcal{H}$, if $u(\cdot, x) \geq \frac{\beta}{2}$ for any $x \in [-r, r]$, then

$$\lim_{\substack{n\to\infty \\ x=x_0+nc}} u_n(\theta, x) = \beta(\theta)$$

uniformly for $\theta \in [-\tau, 0]$. Assume for the sake of contradiction that $W(\theta, x - nc)$ is a traveling wave connecting β to 0. Then $W(\cdot, -\infty) = \beta$ implies that there is a point $-h \in \mathcal{H}$ such that $W(\cdot, x) \geq \frac{\beta}{2}$ for any $x \leq -h$. By hypothesis (A1), we see that $V(\theta, x) := W(\theta, x - h - r)$ is also a traveling wave profile. Moreover, $V(\cdot, x) = W(\cdot, x - h - r) > \frac{\beta}{2}$ for $x \in [-r, r]$. Since $V(\cdot, +\infty) = 0$, there is $x_0 \in \mathcal{H}$ such that $V(\cdot, x_0) < \beta$. Hence, we have

$$\lim_{n\to\infty} T_{-nc}[Q^n[V]](\theta, x_0) = \lim_{\substack{n\to\infty \\ x=x_0+nc}} Q^n[V](\theta, x) = \beta(\theta)$$

uniformly for $\theta \in [-\tau, 0]$. But $T_{-nc}[Q^n[V]](\cdot, x_0) = V(\cdot, x_0) < \beta$, which is a contradiction. □

In order to obtain the existence of the traveling wave with the wave speed $c \geq c^*$, we need to strengthen hypothesis (A3) into the following one:

(A6) One of the following two conditions holds:
   (a) $Q[\mathcal{C}_\beta]$ is precompact in $\mathcal{C}_\beta$.
   (b) There exists a nonnegative number $\varsigma < \tau$ such that $Q[u](\theta, x) = u(\theta + \varsigma, x)$ for $-\tau \le \theta < -\varsigma$, the operator
$$S[u](\theta, x) := \begin{cases} u(0, x), & -\tau \le \theta < -\varsigma, \\ Q[u](\theta, x), & -\varsigma \le \theta \le 0, \end{cases}$$
   is continuous on $\mathcal{C}_\beta$, and $S[\mathcal{C}_\beta]$ is precompact in $\mathcal{C}_\beta$.

We remark that if $\mathcal{H}$ is discrete, then hypothesis (A3) on Q implies hypothesis (A6). Moreover, if (A6)(b) holds and there is an integer n such that $n\varsigma \ge \tau$, then $\{Q^n[u] : u \in \mathcal{C}_\beta\}$ is precompact in $\mathcal{C}_\beta$.

THEOREM 4.2 *Let Q satisfy* (A1)–(A6)*, and let c^* be its asymptotic speed of spread. Then for any $c \ge c^*$, Q has a traveling wave $W(\theta, x - nc)$ connecting β to* 0 *such that $W(\theta, x)$ is nonincreasing in x. Moreover, if $\mathcal{H} = \mathbb{R}$, then $W(\theta, x)$ is continuous in* (θ, x).

PROOF: Let $\tilde{\mathcal{C}}_\beta$ and $\tilde{Q}$ be defined as in Lemma 2.1, and let $\phi \in \tilde{\mathcal{C}}_\beta$ be fixed such that (B1)–(B3) hold. For any number $\kappa \in (0, 1]$, we define an operator $R_{c,\kappa}$ by
$$R_{c,\kappa}[a](\theta, s) := \max\{\kappa\phi(\theta, s), T_{-c}[\tilde{Q}[a]](\theta, s)\},$$
and a sequence of vector-valued functions $a_n(c, \kappa; \theta, s)$ of $\theta \in [-\tau, 0]$, $s \in \mathbb{R}$, by the recursion
$$a_0(c, \kappa; \theta, s) = \kappa\phi(\theta, s), \quad a_{n+1}(c, \kappa; \theta, s) = R_{c,\kappa}[a_n(c, \kappa; \cdot)](\theta, s). \tag{4.1}$$
Note that $a(c, \kappa; \theta, s) = \lim_{n\to\infty} a_n(c, \kappa; \theta, s)$ exists pointwise.

Let $c \ge c^*$ be given. We distinguish between two cases.

Case 1. $\mathcal{H}$ is discrete. By hypothesis (A3) of Q and Lemma 2.5, it follows that
$$\{a_n(c, \kappa; \theta, s) : s \in \mathcal{D}_c,\ k \in (0, 1],\ n \ge 0\}$$
is a family of equicontinuous functions in θ. Hence, for each κ, $a(c, \kappa; \theta, s)$ is continuous in θ and nonincreasing in $s \in \mathcal{D}_c \subset \mathbb{R}$. Moreover, $\{a(c, \kappa; \theta, s) : s \in \mathcal{D}_c, \kappa \in (0, 1]\}$ is a family of equicontinuous functions in θ, and
$$a(c, \kappa; \theta, s) = \max\{\kappa\phi(\theta, s), Q[a(c, \kappa; \cdot, \cdot + s + c)](\theta, 0)\}. \tag{4.2}$$
Fix $\theta_0 \in [-\tau, 0]$. For any $l \in \mathcal{H}$, we define the sequence
$$K_\kappa(l) := \frac{1}{2}[a(c, \kappa; \theta_0, l) + a(c, \kappa; \theta_0, l + 1)].$$
Since $\lim_{l\to-\infty} a(c, \kappa; \theta_0, l) = \beta(\theta_0)$ and $\lim_{l\to\infty} a(c, \kappa; \theta_0, l) = 0$, there exists l_κ such that $\beta(\theta_0)/4 \le K_\kappa(l_\kappa) \le 3\beta(\theta_0)/4$.

Now we consider the sequence $a(c, \kappa; \theta, s + l_\kappa)$. Since $\mathcal{D}_c$ has countably many points, we can find a subsequence $\kappa_i \to 0$ such that
$$\lim_{\kappa_i\to 0} a(c, \kappa_i; \theta, s + l_{\kappa_i}) = W(c; \theta, s) \quad \forall s \in \mathcal{D}_c,$$

and the convergence is uniform for $\theta \in [-\tau, 0]$. From (4.2), we see that

$$W(c; \theta, s-(n+1)c) = Q[W(c; \cdot, \cdot + s - nc)](\theta, 0) \quad \forall \theta \in [-\tau, 0],\ s \in \mathcal{D}_c,\ n \geq 0.$$

Note that

$$W(c; \cdot, -\infty) = \lim_{n\to\infty} W(c; \cdot, s-nc) = \lim_{n\to\infty} Q^n[W](\cdot, s) = \beta \quad \forall s \in \mathcal{H}$$

and

$$W(c; \cdot, +\infty) = \lim_{\substack{s\to\infty \\ s\in\mathcal{H}}} W(c; \cdot, s-c) = \lim_{\substack{s\to\infty \\ s\in\mathcal{H}}} Q[W](\cdot, s) = Q[W(c; \cdot, +\infty)].$$

Since

$$W(\theta_0, 1) \leq \lim_{\kappa_i \to 0} K_{\kappa_i}(l_{\kappa_i}) \leq \frac{3\beta(\theta_0)}{4}$$

and $W(c; \theta, s)$ is nonincreasing in s, it follows that $W(c; \cdot, +\infty) = 0$. Therefore, $W(c; \theta, s-nc)$ is a traveling wave with speed c.

Case 2. $\mathcal{H} = \mathbb{R}$. In this case, we only show that $\{a_n(c, \kappa; \theta, s) : n \geq 1,\ \kappa \in (0, 1]\}$ is a family of equicontinuous functions of (θ, s) in any bounded subset of $[-\tau, 0] \times \mathcal{H}$. The rest of the proof is similar to that in the case where $\mathcal{H} = \mathbb{Z}$. Note that $\{a_0(c, \kappa; \theta, s) : \kappa \in (0, 1]\}$ is a family of equicontinuous functions in (θ, s) on $[-\tau, 0] \times \mathcal{H}$; that is, for any $\epsilon > 0$, there is $\delta_0 > 0$ such that

$$|a_0(c, \kappa; \theta_1, s_1) - a_0(c, \kappa; \theta_2, s_2)| < \epsilon$$

whenever $(\theta_1, s_1), (\theta_2, s_2) \in [-\tau, 0] \times \mathcal{H}$ and $|(\theta_1, s_1) - (\theta_2, s_2)| < \delta_0$. By hypothesis (A6), it follows that

$$Q[a_0](\theta, x) = \begin{cases} a_0(\theta + \varsigma, x), & -\tau \leq \theta < -\varsigma, \\ S[a_0](\theta, x), & -\varsigma \leq \theta \leq 0, \end{cases}$$

and there is $\delta > 0$ such that $|S[v](c, \kappa; \theta_1, s_1) - S[v](c, \kappa; \theta_2, s_2)| < \epsilon$ whenever $v \in \mathcal{C}_\beta$, $(\theta_1, s_1), (\theta_2, s_2) \in [-\tau, 0] \times \mathcal{H}$, and $|(\theta_1, s_1) - (\theta_2, s_2)| < \delta$. This implies that $Q[a_0](\theta_1, x_1) - Q[a_0](\theta_2, x_2) < \epsilon$ whenever $-\varsigma \leq \theta_1, \theta_2 \leq 0$ and $|(\theta_1, x_1) - (\theta_2, x_2)| < \delta$, or $-\tau \leq \theta_1, \theta_2 \leq -\varsigma$ and $|(\theta_1, x_1) - (\theta_2, x_2)| < \delta_0$.

Since $a_1 = \max\{a_0, T_{-c}[Q[a_0]]\}$, we have $|a_1(\theta_1, x) - a_1(\theta_2, x)| < \epsilon$ whenever $-\varsigma \leq \theta_1, \theta_2 \leq 0$ and $|(\theta_1, x_1) - (\theta_2, x_2)| < \delta_1 := \min\{\delta, \delta_0\}$, or $-\tau \leq \theta_1, \theta_2 \leq -\varsigma$ and $|(\theta_1, x_1) - (\theta_2, x_2)| < \delta_0$. Thus, we obtain that $|a_1(\theta_1, x_1) - a_1(\theta_2, x_2)| < 2\epsilon$ whenever $-\tau \leq \theta_1, \theta_2 \leq 0$ and $|(\theta_1, x_1) - (\theta_2, x_2)| < \min\{\delta_1, \delta_0\} = \delta_1$. By the same argument as in the proof of Lemma 2.5(iv), it then follows that $\{a_n(c, \kappa; \theta, s) : n \geq 1,\ \kappa \in (0, 1]\}$ is a family of equicontinuous functions of (θ, s) in any bounded subset of $[-\tau, 0] \times \mathcal{H}$. □

In the rest of this section, we consider traveling waves for the continuous-time semiflow $\{Q_t\}_{t=0}^{\infty}$ on $\mathcal{C}_\beta$. We say that $W(\theta, x - ct)$ is a *traveling wave* of $\{Q_t\}_{t=0}^{\infty}$ if $W : [-\tau, 0] \times \mathbb{R} \to \mathbb{R}^k$ and $Q_t[W](\theta, x) = W(\theta, x - tc)$, and $W(\theta, x - ct)$ *connects* β *to* 0 if $W(\,\cdot\,, -\infty) = \beta$ and $W(\,\cdot\,, +\infty) = 0$.

The following result is a straightforward consequence of Theorem 4.1.

THEOREM 4.3 *Suppose that $Q = Q_1$ satisfies hypotheses* (A1)–(A5)*, and let c^* be the asymptotic speed of spread of Q_1. Then for any $0 < c < c^*$, $\{Q_t\}_{t=0}^{\infty}$ has no traveling wave $W(\theta, x - ct)$ connecting β to* 0*.*

THEOREM 4.4 *Suppose that for any $t > 0$, Q_t satisfies hypotheses* (A1)–(A6)*, and let c^* be the asymptotic speed of spread of Q_1. Then for any $c \geq c^*$, $\{Q_t\}_{t=0}^{\infty}$ has a traveling wave $W(\theta, x - ct)$ connecting β to* 0 *such that $W(\theta, s)$ is continuous and nonincreasing in $s \in \mathbb{R}$.*

PROOF: By Theorem 2.17, it follows that for each $t > 0$, tc^* is the asymptotic speed of spread of the map Q_t. Let $c \geq c^*$ be fixed. In the case where $\mathcal{H} = \mathbb{R}$, the proof is similar to that of [20, theorem 4.1]. Suppose that $W_t(\theta, x - ntc)$ is the traveling wave of Q_t. First, we prove the equicontinuity of $\{W_t\}$. Note that

$$W_t(\theta, x) = T_{-ntc}[Q_{nt}[W_t]](\theta, x) = Q_{nt}[T_{-ntc}[W_t]](\theta, x).$$

For any $t > 0$, there is an integer n such that $nt > 2\tau$, and

$$W_t(\theta, x) = T_{-ntc}[Q_{nt}[W_t]](\theta, x) = Q_{2\tau}[Q_{nt-2\tau}[T_{-ntc}[W_t]]](\theta, x).$$

By assumption (A6), $Q_{2\tau}[\mathcal{C}_\beta]$ is a family of equicontinuous functions, and so is $\{W_t : t > 0\}$. Moreover, we can choose W_t such that $W_t^i(\theta_0, 0) = \beta^i(\theta_0)$. Thus, there is a sequence of integers $r_i \to \infty$ such that $W_{2^{-r_i}}$ converges to W with respect to the compact open topology. Since $W_{2^{-r_i}}$ is a traveling wave profile for all Q_t for which t is a multiple of 2^{-r_i}, $Q_t[W](\theta, x) = W(\theta, x - ct)$ for every fraction t whose denominator is a power of 2. Let t be an arbitrary positive number, and m be any positive integer. Then t can be written as $t = k_m 2^{-m} - r_m$, where k_m is a positive integer and $0 \leq r_m < 2^{-m}$. Thus, we have

$$\begin{aligned} Q_t[W](\theta, x) - W(\theta, x - ct) &= (Q_t[W](\theta, x) - Q_{r_m}[Q_t[W]](\theta, x)) \\ &\quad + (W(\theta, x - c(t + r_m)) - W(x - ct)). \end{aligned}$$

Note that $r_m \to 0$ as $m \to \infty$. By the continuity of W and the fact that $Q_{r_m}[v] \to v$ for any v, it follows that $Q_t[W](\theta, x) = W(\theta, x - ct)$ for any $t \geq 0$. Moreover, since $W_{2^{-r_i}}(\theta, x)$ are nonincreasing in x, so is W. Since $Q_t[W](\theta, x) = W(\theta, x - ct)$, we obtain

$$Q_t[W](\theta, -\infty) = W(\theta, -\infty), \quad Q_t[W](\theta, +\infty) = W(\theta, +\infty).$$

In view of

$$W^i(\theta_0, -\infty) \geq W^i(\theta_0, 0) = W_t^i(\theta_0, 0) = \beta^i(\theta_0) \geq W(\theta_0, +\infty),$$

we see that $W(\,\cdot\,, -\infty) = \beta$ and $W(\,\cdot\,, +\infty) = 0$.

Next we consider the case where $\mathcal{H} = \mathbb{Z}$. For any nonnegative integer r, let $t_r = 2^{-r}/c$. Then each Q_{t_r} has a traveling wave $W_r(\theta, x - n \cdot 2^{-r})$ on the set $[-\tau, 0] \times D_r$ with $D_r = \{x - n2^{-r} : x \in \mathbb{Z}, n \in \mathbb{N}\}$. Let $D = \bigcup_{r=0}^{\infty} D_r$. Since D is a countable set and for each $x \in D$, $x \in D_r$ for all sufficiently large r, we can find a subsequence $r_i \to \infty$ such that $W_{r_i}(\theta, x)$ converges to $W(\theta, x)$ uniformly for $\theta \in [-\tau, 0]$, and $W \not\equiv \beta, 0$, $W(\,\cdot\,, -\infty) = \beta$, $W(\,\cdot\,, +\infty) = 0$. Since $W_{r_i}(\theta, x)$ is nonincreasing in x, so is W. Note that if $r_i \geq r$, then $Q_{nt_r}[W_{r_i}](\theta, x) = W_{r_i}(\theta, x - n2^{-r})$, and

$$Q_{nt_r}[W](\theta, x) = W(\theta, x - n2^{-r}) \quad \forall x \in D,\ n \geq 0,\ r \in \mathbb{Z}. \tag{4.3}$$

For any $x \in D$, let $U_x(\theta, s) := Q_{x/c - s/c}[W](\theta, x)$. We claim that U_x does not depend on x. In fact, (4.3) implies that $Q_{d/c}[W](\theta, x + d) = W(\theta, x)$. Thus, we have

$$\begin{aligned} U_{x+d}(\theta, s) &= Q_{(x+d)/c - s/c}[W](\theta, x + d) \\ &= Q_{x/c - s/c}[Q_{d/c}[W]](\theta, x + d) \\ &= T_{-d}[Q_{x/c - s/c}[Q_{d/c}[W]]](\theta, x) \\ &= Q_{x/c - s/c}[T_{-d}[Q_{d/c}[W]]](\theta, x) \\ &= Q_{x/c - s/c}[W](\theta, x) \\ &= U_x(\theta, s) \end{aligned}$$

for all $d \in D$. Define $U(\theta, s) := U_x(\theta, s)$. Then

$$U(\theta, x - ct) = Q_t[W](\theta, x) = W(\theta, x - ct) \quad \forall x \in D,\ ct \in D.$$

Note that $U(\theta, x) = W(\theta, x)$ $\forall x \in D$. Since D is dense in $\mathbb{R}$ and W is nonincreasing on D, it follows that $U(\theta, s)$ is also nonincreasing in $s \in \mathbb{R}$. Hence, $W(\theta, x - ct) = U(\theta, x - ct)$ is a continuous traveling wave connecting β to 0. □

We conclude our presentation of the theory of spreading speeds and traveling waves with a general remark, which will be used in the next section and may be of its own interest.

Remark 4.5. All results in Sections 2 through 4 are still valid provided that the interval $[-\tau, 0]$ is replaced with a compact metric space and that hypotheses (A3) and (A6) are replaced with (A3)(a) and (A6)(a), respectively.

5 Applications

In this section, we apply the results in Sections 2 through 4 to a functional differential equation with diffusion, a nonlocal and time-delayed lattice differential system, and a reaction-diffusion equation in a cylinder.

5.1 A Functional Differential Equation with Diffusion

Let $\tau > 0$ be fixed and $\bar{\mathcal{C}} := C([-\tau, 0], \mathbb{R})$. We consider a general autonomous functional differential equation with diffusion on $\mathbb{R}$

$$\frac{\partial u(t,x)}{\partial t} = d\,\frac{\partial^2 u(t,x)}{\partial x^2} + f(u_t(\cdot, x)), \quad t > 0,\ x \in \mathbb{R}, \tag{5.1}$$

where $d > 0$, $f : \bar{\mathcal{C}} \to \mathbb{R}$ is a C^1-functional, and for each $x \in \mathbb{R}$, $u_t(\cdot, x)$ denotes the member of $\bar{\mathcal{C}}$ defined by

$$u_t(\theta, x) = u(t+\theta, x) \quad \forall \theta \in [-\tau, 0].$$

To get concrete examples of (5.1), we need to specify the functional f. For example, letting $f(\phi) = F(\phi(0), \phi(-r_1), \phi(-r_2), \dots, \phi(-r_m))$ with all $r_i \geq 0$ and $\tau = \max_{1\leq i\leq m}\{r_i\}$, we obtain a local reaction-diffusion equation with finitely many delays,

$$\begin{aligned}\frac{\partial u(t,x)}{\partial t} = {} & d\frac{\partial^2 u(t,x)}{\partial x^2} \\ & + F(u(t,x), u(t-r_1,x), u(t-r_2,x), \dots, u(t-r_m,x));\end{aligned} \tag{5.2}$$

letting $f(\phi) = F(\phi(0)) + \int_{-\tau}^{0} K(s)G(\phi(s))ds$, we have a local reaction-diffusion equation with distributed delays

$$\begin{aligned}\frac{\partial u(t,x)}{\partial t} &= d\,\frac{\partial^2 u(t,x)}{\partial x^2} + F(u(t,x)) + \int_{-\tau}^{0} K(s)G(u(t+s,x))ds \\ &= d\,\frac{\partial^2 u(t,x)}{\partial x^2} + F(u(t,x)) + \int_{t-\tau}^{t} K(s-t)G(u(s,x))ds.\end{aligned} \tag{5.3}$$

For any $u \in \mathbb{R}$, we write $\hat{u}$ for the element of $\bar{\mathcal{C}}$ satisfying $\hat{u}(\theta) \equiv u$, and define the function $\hat{f} : \mathbb{R} \to \mathbb{R}$ by $\hat{f}(u) = f(\hat{u})$. We need the following assumptions on f to study the spreading speed and traveling waves for (5.1):

- (F1) $\hat{f}(0) = \hat{f}(\beta) = 0$ for some constant $\beta > 0$, $\hat{f}$ has no zero in $(0, \beta)$, and $\hat{f}'(0) > 0$.
- (F2) For each $\phi \in \bar{\mathcal{C}}_\beta$, the derivative $L(\phi) := df(\phi)$ of f can be represented as

$$L(\phi)\psi = a(\phi)\psi(0) + \int_{-\tau}^{0} \psi(\theta) d_\theta \eta(\phi) := a(\phi)\psi(0) + \bar{L}(\phi)\psi,$$

  where $\eta(\phi)$ is a positive Borel measure on $[-\tau, 0]$, $\bar{L}(\phi)\psi \geq 0$ whenever $\psi \geq 0$, and $\eta(\phi)([-\tau, -\tau+\epsilon)) > 0$ for all small $\epsilon > 0$.

By [36, lemma 5.3.3], f is quasi-monotone on $\bar{\mathcal{C}}_\beta$ in the sense that $f(\phi) \leq f(\psi)$ whenever $\phi \leq \psi$ in $\bar{\mathcal{C}}_\beta$ and $\phi(0) = \psi(0)$. Using the semigroup generated by the heat equation and [25, cor. 5] (see, e.g., the proof of [37, theorem 2.2]), we can show that (5.1) generates a monotone semiflow $Q_t : \mathcal{C}_\beta \to \mathcal{C}_\beta$ defined by

$$Q_t(\phi)(\theta, x) = u_t(\theta, x, \phi) \quad \forall (\theta, x) \in [-\tau, 0] \times \mathbb{R},$$

where $u(t, x, \phi)$ is the unique solution of (5.1) satisfying $u_0(\cdot, \cdot, \phi) = \phi \in \mathcal{C}_\beta$. Let $\hat{Q}_t$ be the restriction of Q_t to $\bar{\mathcal{C}}_\beta$. It is easy to see that $\hat{Q}_t : \bar{\mathcal{C}}_\beta \to \bar{\mathcal{C}}_\beta$ is the solution semiflow generated by the following functional differential equation:

$$\frac{du(t)}{dt} = f(u_t), \quad t \geq 0, \tag{5.4}$$

with initial data $u_0 = \phi \in \bar{\mathcal{C}}_\beta$. By [36, cor. 5.3.5], $\hat{Q}_t$ is eventually strongly monotone on $\bar{\mathcal{C}}_\beta$. Moreover, the assumption that $\hat{f}'(0) > 0$ implies that $\hat{0}$ is an unstable equilibrium of (5.4) (see [36, cor. 5.5.2]). By the Dancer-Hess connecting-orbit lemma (see, e.g., [52, p. 39]), the semiflow $\hat{Q}_t$ admits a strongly monotone full orbit connecting 0 to β. Thus, assumption (A5) holds for each map Q_t, $t > 0$.

Define the linear operator $L(t) : \mathcal{C} \to \mathcal{C}$, $t \geq 0$, by the relation

$$L(t)\phi(\theta, x) = \begin{cases} \phi(t+\theta, x) - \phi(0, x), & t+\theta < 0,\ x \in \mathbb{R}, \\ 0, & t+\theta \geq 0,\ -\tau \leq \theta \leq 0,\ x \in \mathbb{R}. \end{cases}$$

Clearly, $L(t) = 0$ for $t \geq \tau$. Define $S(t) := Q_t - L(t)$, $t \geq 0$. By the smoothing property of the semigroup associated with the heat equation, it then follows that Q_t satisfies (A6)(a) for $t \geq \tau$, and (A6)(b) with $\varsigma = t$ for $t \in (0, \tau)$ (see also the proof of [16, theorem 6.1]). Now it is easy to see that for each $t > 0$, the solution map Q_t of (5.1) satisfies all assumptions (A1)–(A6). By Theorems 2.17, 4.3, and 4.4, we then have the following result:

THEOREM 5.1 *Let* (F1) *and* (F2) *hold, and let c^* be the asymptotic speed of spread of the solution map Q_1 of* (5.1) *. Then the following statements are valid*:

(i) *For any $c > c^*$, if $\phi \in \mathcal{C}_\beta$ with $0 \leq \phi \ll \beta$ and $\phi(\cdot, x) = 0$ for x outside a bounded interval, then* $\lim_{t\to\infty, |x|\geq tc} u(t, x, \phi) = 0$.

(ii) *For any $c < c^*$ and $\sigma \in \bar{\mathcal{C}}_\beta$ with $\sigma \gg 0$, there is a positive number r_σ such that if $\phi \in \mathcal{C}_\beta$ and $\phi(\cdot, x) \gg \sigma$ for x on an interval of length $2r_\sigma$, then* $\lim_{t\to\infty, |x|\leq tc} u(t, x, \phi) = \beta$. *If, in addition, f is subhomogeneous on $\mathcal{C}_\beta$, then r_σ can be chosen to be independent of $\sigma \gg 0$.*

(iii) *For any $c \geq c^*$,* (5.1) *has a traveling wave solution $U(x - ct)$ such that $U(s)$ is continuous and nonincreasing in $s \in \mathbb{R}$, $U(-\infty) = \beta$, and $U(+\infty) = 0$. Moreover, for any $c < c^*$,* (5.1) *has no traveling wave $U(x - ct)$ connecting β to* 0.

In order to estimate the spreading speed c^*, we impose the following additional condition on f:

(F3) $f(\phi) \leq L\phi := L(\hat{0})\phi$ for all $\phi \in \bar{\mathcal{C}}_\beta$, and for any $\epsilon \in (0, 1)$, there exists $\delta \in (0, \beta)$ such that $f(\phi) \geq L_\epsilon\phi := a(\hat{0})\phi(0) + (1-\epsilon)\bar{L}(\hat{0})\phi$ for all $\phi \in \bar{\mathcal{C}}_\delta$.

Let $v(t, \phi)$ be the solution of the linear functional differential equation

$$\frac{dv(t)}{dt} = d\mu^2 v(t) + Lv_t \tag{5.5}$$

satisfying $v_0 = \phi \in \bar{\mathcal{C}}$. It is easy to see that $u(t,x) = e^{-\mu x} v(t,\phi)$ is the solution of the linear functional differential equation with diffusion

$$\frac{\partial u}{\partial t} = d\frac{\partial^2 u(t,x)}{\partial x^2} + Lu_t(\cdot\,, x). \tag{5.6}$$

Let M_t be the solution map associated with (5.6), and B_μ^t be defined by M_t as in Section 3. By the above observation, it is easy to see that B_μ^t is just the solution map of the linear functional differential equation (5.5) on $\bar{\mathcal{C}}$. Since (5.5) is a cooperative and irreducible delay equation, it follows that its characteristic equation admits a real root $\lambda = \lambda(\mu)$ that is greater than the real parts of all other roots (see [36, theorem 5.5.1]). Define $\psi \in \bar{\mathcal{C}}$ by $\psi(\theta) := e^{\lambda\theta}$ $\forall\theta \in [-\tau, 0]$. Clearly, $v(t,\psi) = e^{\lambda t}$ $\forall t \geq 0$. Then we have

$$B_\mu^t(\psi) = v(t + \cdot, \psi) = e^{\lambda t}\psi \quad \forall t \geq 0.$$

Thus, $e^{\lambda t}$ is the principal eigenvalue of B_μ^t with positive eigenfunction ψ. Evidently a similar analysis can be made for L_ϵ. By Theorem 3.10, it then follows that the spreading speed of the solution map Q_1 is $c^* = \inf_{\mu>0} \lambda(\mu)/\mu$ provided that assumptions (F1)–(F3) hold.

We remark that the theory developed in Sections 2 through 4 can also be employed to study the spreading speeds and traveling waves for both systems of functional differential equations with diffusions and nonlocal reaction-diffusion equations with time delays. For an integral-equations approach to scalar nonlocal and delayed reaction-diffusion equations, we refer to [40].

5.2 A Nonlocal Lattice Differential System

We consider a nonlocal and time-delayed lattice differential system

$$\begin{aligned} \frac{dw_j(t)}{dt} &= D[w_{j+1}(t) + w_{j-1}(t) - 2w_j(t)] - dw_j(t) \\ &\quad + \frac{\mu}{2\pi}\sum_{k=-\infty}^{\infty} \beta_\alpha(j-k)b(w_k(t-r)), \quad t > 0,\ j \in \mathbb{Z}, \end{aligned} \tag{5.7}$$

where

$$\beta_\alpha(l) = 2e^{-\nu}\int_0^\pi \cos(l\omega)e^{\nu\cos\omega}\,d\omega \tag{5.8}$$

and D, d, μ, and $\nu = 2\alpha$ are all positive real numbers. Moreover, the continuous function $b : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies the following conditions:

(D1) $b(0) = 0$, $b'(0) > \frac{d}{\mu}$, and $b(w) \leq b'(0)w$ for $w \in \mathbb{R}_+$.

(D2) $b(\cdot)$ is strictly increasing on $[0, K]$ for some $K > 0$, and $\mu b(w) = dw$ has a unique solution $w^+ \in (0, K]$.

System (5.7) was derived in [48] to model the growth of a single mature population. By [48, lemma 2.1], we have the following conclusions:

(1) $\frac{1}{2\pi}\sum_{l=-\infty}^{\infty}\beta_\alpha(l) = 1$ and $\beta_\alpha(l) \geq 0$ for all l.

(2) Let $\mathcal{H} = \mathbb{Z}$ and $\tau = r$, and let the set $\mathcal{C}_{w^+}$ be defined as in Section 2. For any $v \in \mathcal{C}_{w^+}$, system (5.7) has a unique global solution $w(t, v) = (w_i(t))_{i=-\infty}^{\infty}$ with $w_i(\theta) := w(\theta, i) = v(\theta, i)\ \forall i \in \mathbb{Z}$, $\theta \in [-r, 0]$, and $0 \le w_i(t) \le w^+\ \forall i \in \mathbb{Z}$, $t \ge 0$.

(3) Let v and $\bar{v}$ be two solutions of (5.7) with $\bar{v}_i(\theta) \le v_i(\theta)$ for all $\theta \in [-r, 0]$, $i \in \mathbb{Z}$. Then $\bar{v}_i(t) \le v_i(t)$ for all $t > 0$, $i \in \mathbb{Z}$.

Note that if w is a solution of

$$\frac{dw(t)}{dt} = -dw(t) + \mu b(w(t-r)), \tag{5.9}$$

then $w_i = w$, $i \in \mathbb{Z}$, is a solution of (5.7). Moreover, if $\bar{v}$ and v are two solutions of (5.9) with $0 \le \bar{v}(\theta) \le v(\theta) \le w^+\ \forall \theta \in [-r, 0]$ and $\bar{v}(\theta_0) < v(\theta_0)$ for some $\theta_0 \in [-r, 0]$, then $\bar{v}(t) < v(t)$ for $t \ge r$ (see [36, theorem 5.3.4]).

Let Q_t be the solution map at time $t \ge 0$ of system (5.7), that is,

$$Q_t(v)(\theta) = w(t+\theta, v) \quad \forall \theta \in [-r, 0],\ v \in \mathcal{C}_{w^+}.$$

Define the linear operator $L(t) : \mathcal{C}_{w^+} \to \mathcal{C}_{w^+}$, $t \ge 0$, by the relation

$$L(t)v(\theta) = \begin{cases} v(t+\theta) - v(0), & t+\theta < 0, \\ 0, & t+\theta \ge 0,\ -\tau \le \theta \le 0. \end{cases}$$

Clearly, $L(t) = 0$ for $t \ge \tau$. We further have the following result on Q_t:

PROPOSITION 5.2 *For each $t > 0$, Q_t satisfies hypotheses* (A1)–(A5). *Moreover, $\{Q_t\}_{t=0}^{\infty}$ is a semiflow on $\mathcal{C}_{w^+}$.*

PROOF: Define $S(t) := Q_t - L(t)$, $t \ge 0$. It then follows that Q_t satisfies (A3)(a) for $t \ge \tau$ and (A3)(b) with $\varsigma = t$ for $t \in (0, \tau)$ (see, e.g., the proof of [16, theorem 6.1]). We prove only the continuity of $Q_t(v) = Q(t, v)$ in (t, v) since all the other conditions are easily verified. Let $v(t)$ and $\bar{v}(t)$ be two solutions of (5.7) with $0 \le v(t), \bar{v}(t) \le w^+$. In order to prove the continuity of $\{Q_t\}_{t=0}^{\infty}$, we first prove the following claim:

Claim. For any $\epsilon > 0$ and $t_0 > 0$, there exist $\delta > 0$ and an integer N such that $|v_0(t) - \bar{v}_0(t)| \le \epsilon\ \forall t \in [0, t_0]$ whenever $|v_i(t) - \bar{v}_i(t)| < \delta$ for $t \in [-r, 0]$, $-N \le i \le N$.

We first consider the case where $v(t) \ge \bar{v}(t)$ for $t \in [-r, 0]$. In this case, we have $v(t) \ge \bar{v}(t)$ for all $t \ge -r$. Let $w = v(t) - \bar{v}(t)$. Then

$$\begin{aligned} \frac{dw_j(t)}{dt} &= D[w_{j+1}(t) + w_{j-1}(t) - 2w_j(t)] - dw_j(t) \\ &\quad + \frac{\mu}{2\pi} \sum_{k=-\infty}^{\infty} \beta_\alpha(j-k)\big(b(v_k(t-r)) - b(\bar{v}_k(t-r))\big). \end{aligned}$$

Thus, there is $L > 0$ such that

$$\frac{dw_j(t)}{dt} \le D[w_{j+1}(t) + w_{j-1}(t) - 2w_j(t)] - dw_j(t) + L \sum_{k=-\infty}^{\infty} \beta_\alpha(j-k) w_k(t-r).$$

In what follows, we divide the proof into two cases: $r > 0$ and $r = 0$. If $r > 0$, then for any $\epsilon > 0$, there are $\delta > 0$ and an integer N such that if $|w_i(t)| < \delta$ for $t \in [-r, 0]$, $-N \le i \le N$, then

$$L \sum_{k=-\infty}^{\infty} \beta_\alpha(j-k) w_k(t-r) \le \epsilon$$

for any $t \in [0, r]$, and hence

$$\frac{dw_j(t)}{dt} \le D[w_{j+1}(t) + w_{j-1}(t) - 2w_j(t)] - dw_j(t) + \epsilon \quad \forall t \in [0, r].$$

It follows that

$$w_j(t) \le \frac{1}{2\pi} e^{-dt} \sum_{k=-\infty}^{\infty} \beta_{Dt}(j-k) w_k(0) - \frac{\epsilon e^{-dt}}{d} + \frac{\epsilon}{d} \quad \forall t \in [0, r].$$

By Dini's theorem, for any ϵ there is an integer N such that

$$\sum_{k=-\infty}^{-N} \beta_{Dt}(k) + \sum_{k=N}^{\infty} \beta_{Dt}(k) \le \frac{\pi\epsilon}{w^+} \quad \forall t \in [0, r].$$

Suppose that $w_i(t) = v_i(t) - \bar{v}_i(t) < \frac{\epsilon}{2(2N+1)}$ $\forall t \in [-r, 0]$, $-N \le i \le N$. We then have

$$\begin{aligned}
&\frac{1}{2\pi} e^{-dt} \sum_{k=-\infty}^{\infty} \beta_{Dt}(k) w_k(0) \\
&= \frac{1}{2\pi} e^{-dt} \left(\sum_{k=-\infty}^{-N} \beta_{Dt}(k) w_k(0) + \sum_{k=-N}^{N} \beta_{Dt}(k) w_k(0) + \sum_{k=N}^{\infty} \beta_{Dt}(k) w_k(0) \right) \\
&\le \frac{1}{2\pi} e^{-dt} \left(\sum_{k=-\infty}^{-N} \beta_{Dt}(k) w^+ + 2\pi \sum_{k=-N}^{N} w_k(0) + \sum_{k=N}^{\infty} \beta_{Dt}(k) w^+ \right) \\
&\le \epsilon.
\end{aligned}$$

Thus,

$$w_0(t) \le \epsilon + \frac{\epsilon}{d} - \frac{\epsilon e^{-dr}}{d} \quad \forall t \in [0, r],$$

which implies that our claim holds in the case where $r > 0$, $t_0 = r$, and $v(t) \ge \bar{v}(t)$.

If $r > 0$, $t_0 = r$, but $v(t) \not\geq \bar{v}(t)$ for $t \in [-r, 0]$, we let $\hat{v}(t), \tilde{v}(t)$ be two solutions of (5.7) with $\hat{v}(t) = \max\{v(t), \bar{v}(t)\}$ and $\tilde{v}(t) = \min\{v(t), \bar{v}(t)\}$ for $t \in [-r, 0]$. Thus, $\tilde{v}(t) \leq v(t), \bar{v}(t) \leq \hat{v}(t)$ for $t \geq r$. Hence, $|v_i(t) - \bar{v}_i(t)| \leq |\hat{v}_i(t) - \tilde{v}_i(t)|\ \forall i \in \mathbb{Z}, t \geq r$. This proves the claim above.

For any $t \in [nr, (n+1)r]$, we have $Q_t = Q_{t-nr}Q_{nr}$. Thus, Q_t is uniformly continuous for $t \in [nr, (n+1)r]$, which implies that Q_t is uniformly continuous for t on any bounded interval. It follows that $Q_t(v)$ is continuous in $(t, v) \in \mathbb{R}_+ \times \mathcal{C}_{w^+}$.

Next we consider the case where $r = 0$. By the discrete Fourier transform, as applied to the linear equation

$$\frac{dw_j(t)}{dt} = D[w_{j+1}(t) + w_{j-1}(t) - 2w_j(t)] - dw_j(t) + L\sum_{k=-\infty}^{\infty} \beta_\alpha(j-k)w_k(t)$$

with the initial value $w_k(0)$ (see, e.g., [48]), we obtain

$$w_j(t) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\left(\int_{-\pi}^{\pi} e^{\mathrm{i}(j-k)\omega + f(\omega)t}\, d\omega\right)w_k(0),$$

where

$$f(\omega) = D(2\cos\omega - 2) - d + L\sum_{k=-\infty}^{\infty}\beta_\alpha(k)\cos(k\omega).$$

As argued for the case $r > 0$, we see that for any $\epsilon > 0$ and $t_0 > 0$, there exist $\delta > 0$ and an integer N such that if $|w_i(0)| < \delta$ for $-N \leq i \leq N$, then $w_0(t) \leq \epsilon$ on $[0, t_0]$. Thus, Q_t is uniformly continuous for t on any bounded interval, and hence $Q_t(v)$ is continuous in $(t, v) \in \mathbb{R}_+ \times \mathcal{C}_{w^+}$. □

Consider the linearized equation of (5.7) at $w = 0$,

$$\frac{dw_j(t)}{dt} = D[w_{j+1}(t) + w_{j-1}(t) - 2w_j(t)] - dw_j(t) + \frac{\mu}{2\pi}b'(0)\sum_{k=-\infty}^{\infty}\beta_\alpha(j-k)w_k(t-r). \tag{5.10}$$

Note that $b'(0)w > b(w)$. It follows that if w is a solution of (5.10), then w is a supersolution of (5.7). Let M_t be the solution map at time t of system (5.10). Then $Q_t[u] \leq M_t[u]$ for any $u \in \mathcal{C}$. Moreover, M_t satisfies the assumptions on M in Section 3.

Now, let us consider the linear system

$$\frac{dw_j(t)}{dt} = D[w_{j+1}(t) + w_{j-1}(t) - 2w_j(t)] - dw_j(t) + \frac{\mu}{2\pi}(1-\epsilon)b'(0)\sum_{k=-\infty}^{\infty}\beta_\alpha(j-k)w_k(t-r) \tag{5.11}$$

with parameter ϵ. For any $\epsilon > 0$, there is a δ such that if $0 \le w < \delta$, then $b(w) > (1-\epsilon)b'(0)w$. Let M_t^ϵ be the solution map at time t of system (5.11). It is easy to see that for any ϵ, there is $\delta' > 0$ such that if $u \in \mathcal{C}$ with $u_i(\theta) < \delta'$ for any $i \in \mathbb{Z}, \theta \in [-r, 0]$, then $Q_t[u] \ge M_t^\epsilon[u]$ for all $t \in [0, 1]$.

For each $\phi \in C([-r, 0], \mathbb{R})$, let $\eta(t, \phi)$ be the unique solution of the linear delay equation

$$\begin{aligned}\frac{d\eta(t)}{dt} &= [D(e^{-\chi}+e^{\chi}) - (d+2D)]\eta(t) \\ &\quad + \frac{\mu}{2\pi}b'(0)\sum_{k=-\infty}^{\infty}\beta_\alpha(j-k)e^{-\chi(j-k)}\eta(t-r)\end{aligned} \tag{5.12}$$

with $\eta(\theta, \phi) = \phi(\theta)\ \forall\theta \in [-r, 0]$. It is easy to see that $w(t) = \{w_j(t)\}_{j=-\infty}^{\infty}$ with $w_j(t) = e^{-\chi j}\eta(t, \phi)$ is a solution of (5.10). Thus, we have

$$B_\chi^t(\phi)(\theta) := M_t[\phi e^{-\chi j}](\theta, 0) = \eta(t+\theta, \phi) \quad \forall\theta \in [-r, 0],$$

which implies that B_χ^t is the solution map at time t of equation (5.12). Note that $\sum_{k=-\infty}^{\infty}\beta_\alpha(j-k)e^{-\chi(j-k)} = 2\pi e^{(\cosh\chi-1)\nu}$ (see [48]). Then (5.12) reduces to

$$\frac{d\eta(t)}{dt} = [D(e^{-\chi}+e^{\chi}) - (d+2D)]\eta(t) + \mu b'(0)e^{(\cosh\chi-1)\nu}\eta(t-r). \tag{5.13}$$

Since (5.13) is a cooperative and irreducible delay equation, it follows that its characteristic equation

$$\lambda - [D(e^{-\chi}+e^{\chi}) - (d+2D)] - \mu b'(0)e^{(\cosh\chi-1)\nu-\lambda r} = 0 \tag{5.14}$$

admits a real root $\lambda = \lambda(\chi)$ that is greater than the real parts of all other roots (see [36, theorem 5.5.1]).

Define $\psi \in C([-r, 0], \mathbb{R})$ by $\psi(\theta) := e^{\lambda\theta}\ \forall\theta \in [-r, 0]$. Clearly $\eta(t, \psi) = e^{\lambda t}$ $\forall t \ge 0$. Then we have

$$B_\chi^t(\psi) = \eta(t+\cdot, \psi) = e^{\lambda t}\psi \quad \forall t \ge 0.$$

Thus, $e^{\lambda t}$ is the principal eigenvalue of B_χ^t with the positive eigenfunction ψ. It is easy to see that $\lambda \ge [D(e^{-\chi}+e^{\chi}) - (d+2D)]$. Then $\Phi(\chi) := \frac{\lambda(\chi)}{\chi}$ assumes its minimum at some finite value χ^*. By Theorem 3.10, it follows that the spreading speed for the continuous-time semiflow $\{Q\}_{t=0}^{\infty}$ is $c^* = \inf\lambda(\chi)/\chi$. Let $c = \Phi(\chi)$. Then $c^* = \Phi(\chi^*)$ and $\frac{dc}{d\chi}\big|_{\chi=\chi^*} = 0$. Define

$$f(c, \chi) := c\chi - [D(e^{-\chi}+e^{\chi}) - (d+2D)] - \mu b'(0)e^{(\cosh\chi-1)\nu-c\chi r}.$$

Consequently, (c^*, χ^*) can be determined as the solution to the system

$$f(c, \chi) = 0, \quad \frac{\partial f}{\partial\chi}(c, \chi) = 0.$$

It is easy to see that if $w(t)$ is a solution of (5.7) with $0 \leq w_i(t) \leq w^+$ for any $t \in [-r, 0]$, $i \in \mathbb{Z}$, and there is some $t_0 \in [-r, 0]$ and i such that $w_i(t_0) > 0$, then $w_i(t) > 0$ for $t > r$ and $i \in \mathbb{Z}$.

As the consequences of Theorem 2.17 with Remark 3.6 and Theorems 4.3 and 4.4, we have the following results:

THEOREM 5.3 *Let $w(t)$ be a solution of* (5.7) *with $0 \leq w_i(t) < w^+$ for any $t \in [-r, 0]$, $i \in \mathbb{Z}$. Then the following statements are valid*:

(i) *If $w_i(t) = 0$ for $t \in [-r, 0]$ and i is outside a bounded interval, then $\lim_{t\to\infty, |i|\geq tc} w_i(t) = 0$ for any $c > c^*$.*
(ii) *If $w(t) \not\equiv 0$ for $t \in [-r, 0]$, then $\lim_{t\to\infty, |i|\leq tc} w_i(t) = w^+$ for any $c < c^*$.*

THEOREM 5.4 *Given any $c \geq c^*$,* (5.7) *has a traveling wave solution $w_i(t) = U(i-tc)$ such that $U(s)$ is continuous and nonincreasing in $s \in \mathbb{R}$, and $U(-\infty) = w^+$ and $U(+\infty) = 0$. Moreover, for any $c < c^*$,* (5.7) *has no traveling wave $U(i - tc)$ connecting w^+ to* 0.

Note that the spreading speed c^* and the existence of traveling waves with wave speed $c > c^*$ were established for system (5.7) in [48]. Our result includes the existence of the traveling wave with wave speed c^* and the nonexistence of traveling waves with wave speed $0 < c < c^*$, which shows that the spreading speed c^* is just the minimal wave speed for monotone traveling waves.

We remark that monotone traveling waves in the monostable case have been studied for the discrete Fisher's equation [53], discrete quasi-linear equations (see, e.g., [8, 9]), and lattice delay differential equations (see e.g., [49]). The asymptotic speeds of spread of these lattice equations can be established by appealing to the theory developed in Sections 2 through 4. In particular, it can be shown that the spreading speed coincides with the minimal wave speed under appropriate conditions.

5.3 A Reaction-Diffusion Equation in a Cylinder

We consider a reaction-diffusion equation in a cylinder

$$(5.15)\quad \begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \Delta_y u + ug(y, u), & x \in \mathbb{R},\ y = (y_1, \dots, y_m) \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \mathbb{R} \times \partial\Omega \times (0, +\infty), \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^m$ with smooth boundary $\partial\Omega$,

$$\Delta_y = \sum_{i=1}^{m} \frac{\partial^2}{\partial y_i^2},$$

and ν is the outer unit normal vector to $\partial\Omega \times \mathbb{R}$. Assume that

(G) $g \in C^1(\overline{\Omega} \times \mathbb{R}_+, \mathbb{R})$, $\frac{\partial g}{\partial u} < 0\ \forall (y, u) \in \overline{\Omega} \times \mathbb{R}_+$, and there is $K > 0$ such that $g(y, K) \leq 0\ \forall y \in \overline{\Omega}$.

Let λ_0 be the principal eigenvalue of the elliptic eigenvalue problem

$$\begin{cases} \lambda v = \Delta_y v + v g(y, 0), & y \in \Omega, \\ \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.16}$$

Assume that $\lambda_0 > 0$. By [52, theorem 3.1.5], it then follows that the reaction-diffusion equation

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta_y u + u g(y, u), & y \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0, +\infty), \end{cases} \tag{5.17}$$

admits a unique positive steady state $\beta(y)$. This implies that equation (5.15) has two equilibrium solutions 0 and $\beta(y)$, and there is no other x-independent equilibrium.

Let $\mathcal{C}$ be the set of all bounded and continuous functions from $\mathbb{R} \times \overline{\Omega}$ to $\mathbb{R}$. We consider the linear equation

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \Delta_y u, & x \in \mathbb{R},\ y \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \mathbb{R} \times \partial\Omega \times (0, +\infty). \end{cases} \tag{5.18}$$

Let $G(t, y, w)$ be the Green's function of the equation (see, e.g., [15])

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta_y u, & y \in \Omega, t > 0, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0, +\infty). \end{cases} \tag{5.19}$$

Then it is easy to verify that

$$e^{-\frac{(x-z)^2}{4\pi t}} G(t, y, w)$$

is the Green's function of equation (5.18). That is, the solution of (5.18) with initial value $u(0, \cdot) = \phi(\cdot) \in \mathcal{C}$ can be expressed as

$$u(t, x, y, \phi) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{+\infty} \int_{\Omega} e^{-\frac{(x-z)^2}{4\pi t}} G(t, y, w) \phi(z, w) dw\, dz.$$

Define $T(t)\phi = u(t, \cdot, \phi)\ \forall \phi \in \mathcal{C}$. It then follows that $\{T(t)\}_{t=0}^{\infty}$ is a linear semigroup on the space $\mathcal{C}$ with respect to the compact open topology. For any $a, b \in \mathcal{C}$, define $[a, b]_{\mathcal{C}} := \{\phi \in \mathcal{C} :\ a \le \phi \le b\}$. For any $t > 0$ and $a, b \in \mathcal{C}$, it is easy to verify that $T(t)[a, b]_{\mathcal{C}}$ is a family of equicontinuous functions.

Now we write (5.15) subject to $u(0, \cdot) = \phi \in \mathcal{C}$ as an integral equation

$$u(t, x, y) = T(t)[\phi](x, y) + \int_0^t T(s) f(y, u(t - s, x, y)) ds, \tag{5.20}$$

where $f(y, u) = u g(y, u)$. Using the standard linear semigroup theory (see, e.g., [25, 28]), we see that for any $\phi \in \mathcal{C}_\beta$, (5.15) has a unique solution $u(t, \phi)$ with $u(0, \phi) = \phi$, which exists globally on $[0, +\infty)$. Define $Q_t(\phi) = u(t, \phi)$. With the expression of the semigroup $T(t)$ and (5.20), we can show that $\{Q_t\}_{t=0}^{\infty}$ is a subhomogeneous semiflow on $\mathcal{C}_\beta$. Moreover, Q_t satisfies hypotheses (A1), (A2), (A3)(a), (A4), (A5), and (A6)(a) for each $t > 0$. Hence, $\{Q_t\}_{t=0}^{\infty}$ has a spreading speed c^*.

Let $\{M_t\}_{t=0}^{\infty}$ be the solution semiflow associated with the linear equation

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \Delta_y u + ug(y,0), & x \in \mathbb{R},\ y \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \mathbb{R} \times \partial\Omega \times (0,+\infty). \end{cases} \tag{5.21}$$

Since $g(y,0) \geq g(y,u)$, we have $M_t[\phi] \geq Q_t[\phi]$ for any $\phi \in \mathcal{C}_\beta$. Let M_t^ϵ be the solution semiflow associated with the linear equation

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \Delta_y u + (1-\epsilon)ug(y,0), & x \in \mathbb{R},\ y \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \mathbb{R} \times \partial\Omega \times (0,+\infty). \end{cases} \tag{5.22}$$

Then for any ϵ, there is a $\delta \gg 0$ such that $M_t^\epsilon[\phi] \leq Q_t[\phi]$ for any $\phi \in \mathcal{C}_\delta$ and $t \in [0,1]$.

It is easy to see that if $\eta(t,y)$ is a solution of the linear equation

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta_y u + ug(y,0) + \mu^2 u, & y \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,+\infty), \end{cases} \tag{5.23}$$

then $u(t,x,y) = \eta(t,y)e^{-\mu x}$ is a solution of (5.21).

Let $\lambda(\mu)$ be the principal eigenvalue of the elliptic eigenvalue problem

$$\begin{cases} \lambda u = \Delta_y u + ug(y,0) + \mu^2 u, & y \in \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.24}$$

It follows that $e^{\lambda(\mu)t}$ is the principal eigenvalue of the $B_\mu(t)$, where $B_\mu(t)$ is the solution semiflow associated with (5.23). It is easy to see that $B_\mu(t)[\alpha](y) = M_t[\alpha e^{-\mu x}](y,0)$. Since $\lambda(\mu) = \lambda_0 + \mu^2$, we see that $\Phi(\mu) = \frac{\lambda(\mu)}{\mu} = \mu + \frac{\lambda_0}{\mu}$ assumes its minimum at $\mu^* = \sqrt{\lambda_0}$. Thus, Theorem 3.10 implies that $c^* = 2\sqrt{\lambda_0}$.

Note that if $u(t,x,y)$ is a solution of (5.15) with $0 \leq u(0,x,y) < \beta(y)\ \forall y \in \Omega, x \in \mathbb{R}$, and $u(0,x,y) \not\equiv 0$, then $u(t,x,y) > 0\ \forall t > 0, y \in \Omega, x \in \mathbb{R}$ (see, e.g., the proof of [51, lemma 3.1]).

As the consequences of Theorems 2.17, 4.3, and 4.4 with Remark 4.5, we have the following results:

THEOREM 5.5 *Let $u(t,x,y)$ be a solution of* (5.15) *with $u(0,\cdot) \in \mathcal{C}_\beta$. Then the following two statements are valid*:

(1) *If $u(0,x,y) = 0$ for $y \in \Omega$ and x outside a bounded interval, then for any $c > c^*$, $\lim_{t\to\infty,|x|\geq tc} u(t,x,y) = 0$ uniformly for $y \in \Omega$.*
(2) *If $u(0,x,y) \not\equiv 0$, then for any $c < c^*$, $\lim_{t\to\infty,|x|\leq tc} u(t,x,y) = \beta(y)$ uniformly for $y \in \Omega$.*

THEOREM 5.6 *For any $c \geq c^*$,* (5.15) *has a traveling wave solution $U(x-tc,y)$ such that $U(s,y)$ is nonincreasing in $s \in \mathbb{R}$, and $\lim_{s\to-\infty} U(s,y) = \beta(y)$ and $\lim_{s\to\infty} U(s,y) = 0$ uniformly for $y \in \Omega$. Moreover, for any $c < c^*$,* (5.15) *has no traveling wave $U(x-tc,y)$ connecting $\beta(\cdot)$ to* 0.

We should mention that traveling waves in the monostable case were already studied in [6, 24, 33, 41] for some parabolic equations in cylinders. As illustrated in the above example, it is also possible to use the theory developed above to obtain the asymptotic speeds of spread for these equations.

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