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ORIGINAL ARTICLE

Global Dynamics of a Reaction–Diffusion Model of Zika Virus Transmission with Seasonality

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Abstract

In this paper, we propose a periodic reaction–diffusion model of Zika virus with seasonal and spatial heterogeneous structure in host and vector population. We introduce the basic reproduction ratio R_0 for this model and show that the disease-free periodic solution is globally asymptotically stable if $R_0 \leq 1$, while the system admits a globally asymptotically stable positive periodic solution if $R_0 > 1$. Numerically, we study the Zika transmission in Rio de Janeiro Municipality, Brazil, and investigate the effects of some model parameters on R_0. We find that the neglect of seasonality underestimates the value of R_0 and the maximum carrying capacity affects the spread of Zika virus.

Keywords Zika virus · Seasonality · Reaction–diffusion model · Basic reproduction ratio · Global stability

Mathematics Subject Classification 35K57 · 37N25 · 92D30

1 Introduction

Zika virus is a mosquito-borne flavivirus, and it is primarily transmitted to humans through bites from two key mosquito vector species: *Aedes aegypti* and *Aedes albopictus* mosquitoes. Approximately 80% of people infected with Zika virus do not develop symptoms; 20% of clinically affected people mostly experience mild symptoms, such as fever, rash, conjunctivitis, muscle and joint pain, malaise, and headache (Caminade

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et al. 2017; WHO 2018). There is considerable evidence indicating that Zika virus infection during pregnancy can lead to microcephaly and other congenital abnormalities in the developing fetus and newborn, and can cause pregnancy complications such as fetal loss, stillbirth, and preterm birth (Brasil et al. 2016; Mlakar et al. 2016). It is also linked to the Guillain–Barré syndrome, neuropathy, and myelitis (Cao-Lormeau et al. 2016).

Zika virus was initially isolated from a rhesus monkey in the Zika forest in Uganda in 1947 (Dick et al. 1952). The first human outbreak of Zika was documented in 1954 in Africa. Recently, Zika entered the Americas and the first confirmed case was in northeastern Brazil in May 2015. Since then, it quickly spread to many other countries, including the USA. In 2015, there were a preliminary estimate of 440,000–1,300,000 Zika cases in Brazil and 4,783 suspected cases of microcephaly (Heukelbach et al. 2016). To date, there are 86 countries and territories which have reported evidence of mosquito-transmitted Zika infection (WHO 2018). The rapid spread of Zika worldwide also leads the World Health Organization to announce a Public Health Emergency of International Concern in 2016. However, there is still no vaccine or other modality available to prevent or treat Zika virus infection.

Mathematical modeling has become an important tool used to describe the spread of Zika virus. Gao et al. (2016) proposed an autonomous ODE model to investigate the impact of mosquito-borne and sexual transmission on the spread of Zika virus and performed sensitivity analysis for the basic reproduction number R_0. Olawoyin and Kribs (2018), considered an autonomous ODE model with the sexually and vertically transmitted within vectors and hosts, and found that R_0 is most sensitive to the mosquito biting rate and transmission probability. For vector-borne diseases, due to the spatial structure of density and location of hosts and vectors, and their movements over space, the spatial heterogeneity in abundance and distribution of host and vector populations has a strong impact on the disease spread and persistence (Charron et al. 2013; Lou and Zhao 2011; Neuhauser 2001; Smith et al. 2004). Thus, the inclusion of diffusion in the transmission and control of diseases in a heterogeneous environment is unavoidable. Accordingly, several reaction–diffusion models have been developed to describe the spatial spread of Zika virus (Cai et al. 2019; Fitzgibbon et al. 2017; Magal et al. 2018; Miyaoka et al. 2019).

Weather and climate factors, especially temperature, are known to impact the transmission dynamics of vector-borne diseases, such as Bluetongue virus, West Nile virus, Dengue, and Schistosomiasis (Charron et al. 2013; Lambrechts et al. 2011; Li and Zhao 2019; Li et al. 2020; Zhang and Zhao 2020). The field and laboratory experiments demonstrate that the development, survival, reproduction and biting rates, the transmission and infection probabilities of both *Aedes aegypti* and *Aedes albopictus* are affected by temperature (Brady et al. 2013; Mordecai et al. 2017). Thus, it is more reasonable to incorporate the seasonality into mathematical models of Zika transmission. In Suparit, Wiratsudakul, and Modchang (2018) developed a Zika virus transmission model with a temperature-dependent mosquito biting rate and gave a case study in Bahia.

Fitzgibbon et al. (2017) studied the spatial interaction of the hosts and vectors by using an autonomous reaction–diffusion model and a periodic model with a time-dependent vector breeding rate numerically, but the mathematical analysis of the global dynamics for such models remains unsolved there. Magal et al. (2018) made a detailed analysis of the autonomous reaction–diffusion model proposed in Fitzgibbon et al. (2017) and proved that the basic reproduction ratio R_0 serves as a threshold value for the evolution dynamics of the model. In order to study the impact of spatiotemporal heterogeneities and movements on the spread and persistence of diseases, it is essential to investigate the role of diffusion and seasonality in the transmission of diseases in a heterogeneous environment.

In this paper, we modify the Zika transmission model developed in Fitzgibbon et al. (2017) by accounting for the seasonality. Our purpose is to study the effects of the seasonal and spatial heterogeneities in abundance and distribution of host and vector populations on the Zika transmission dynamics. Mathematically, we give a novel method to prove the global stability for the model by using the theory of chain transitive sets, which enables us to easily lift the threshold type result from a limiting system to the full model. This method can be also applied to the autonomous reaction–diffusion model to greatly simplify the analysis in Magal et al. (2018).

The rest of this paper is organized as follows. In Sect. 2, we present the model and study its well-posedness. In Sect. 3, we first derive the basic reproduction ratio R_0 and then establish the threshold type result on the global stability in terms of R_0. In Sect. 4, we use numerical simulations to reveal the spatiotemporal spread of Zika virus in Rio de Janeiro Municipality, Brazil. A brief discussion then concludes the paper.

2 The Model

We divide the population into two subpopulations, i.e., the host and vector populations. Suppose that all populations are living in a bounded domain $\Omega \in \mathbb{R}^n$ with smooth boundary $\partial\Omega$. For the vector population, we only consider one species, *Aedes aegypti* or *Aedes albopictus* mosquitoes, because generally there is only one primary vector in the Zika outbreak region. All vector populations refer to adult female mosquitoes because only such vectors contract the virus. Let $H_i(t, x)$, $V_u(t, x)$ and $V_i(t, x)$ be the densities of infected hosts, susceptible vectors, and infected vectors at time t and location x, respectively. Here we assume all parameters are temperature dependent. Note that the temperature C can be regarded as a function of time t, that is, $C = C(t)$. The host and vector movements are assumed to be an unbiased random walk, and $\delta_1(t, x)$ and $\delta_2(t, x)$ are the host and vector diffusion rates at time t and location x, respectively. $\lambda(t, x)$ is the loss rate of the infected host population due to the recovery and death rate at time t and location x. In this model, we assume both the susceptible and infectious vectors give birth, and all newborn vectors are susceptible and enter the susceptible class at breeding rate $\beta(t, x)$ at time t and location x. We assume that Zika does not affect the mosquito lifespan, and the natural mortality rate of vector $\mu_1(t, x)$ is the inversely proportional to the vector lifespan. $\mu_2(t, x)$ is the density-dependent loss rate of vector at time t and location x, which is estimated by $(\beta(t, x) - \mu_1(t, x))/N_m(t, x)$, where $N_m(t, x)$ is the maximum carrying capacity for the vector population at time t and location x. $\sigma_1(t, x)$ and $\sigma_2(t, x)$ are the transmission rate for susceptible hosts and vectors, respectively, which is a product of the per capita biting rate $a(t, x)$ of vector on hosts and the transmission probability $\beta_{vh}(t, x)$ ($\beta_{hv}(t, x)$) from infectious vectors to susceptible hosts (from infectious hosts to susceptible vectors) per bite. $H_u(x)$ is the density of susceptible host population depending on the spatial location x. Here we assume that the susceptible host population is unchanging by the epidemic during a relatively short time period since the infected rate is fairly small (Magal et al. 2018). Accordingly, the earlier model in Fitzgibbon et al. (2017) can be modified as

$$
\begin{aligned}
\frac{\partial H_i(t,x)}{\partial t} &= \nabla \cdot (\delta_1(t,x)\nabla H_i(t,x)) - \lambda(t,x)H_i(t,x) \\
&\quad + \sigma_1(t,x)H_u(x)V_i(t,x), t > 0, x \in \Omega, \\
\frac{\partial V_u(t,x)}{\partial t} &= \nabla \cdot (\delta_2(t,x)\nabla V_u(t,x)) - \sigma_2(t,x)V_u(t,x)H_i(t,x) \\
&\quad + \beta(t,x)(V_u(t,x) + V_i(t,x)) - \mu_1(t,x)V_u(t,x) \\
&\quad - \mu_2(t,x)(V_u(t,x) + V_i(t,x))V_u(t,x), t > 0, x \in \Omega, \\
\frac{\partial V_i(t,x)}{\partial t} &= \nabla \cdot (\delta_2(t,x)\nabla V_i(t,x)) + \sigma_2(t,x)V_u(t,x)H_i(t,x) - \mu_1(t,x)V_i(t,x) \\
&\quad - \mu_2(t,x)(V_u(t,x) + V_i(t,x))V_i(t,x), t > 0, x \in \Omega, \\
\frac{\partial H_i}{\partial \nu} &= \frac{\partial V_u}{\partial \nu} = \frac{\partial V_i}{\partial \nu} = 0, t > 0, x \in \partial\Omega,
\end{aligned}
\tag{1}
$$

where ∇ is the gradient operator and ν is the outward normal vector to $\partial\Omega$. We make the following basic assumptions:

(A1) Functions $\beta(t,x) \not\equiv 0$, $\sigma_1(t,x)$ and $\mu_2(t,x)$ are Hölder continuous and nonnegative nontrivial on $\mathbb{R} \times \bar{\Omega}$, and T-periodic in t; functions $\lambda(t,x)$, $\mu_1(t,x)$, $\sigma_2(t,x)$ and the diffusion coefficients $\delta_1(t,x)$ and $\delta_2(t,x)$ are Hölder continuous and positive on $\mathbb{R} \times \bar{\Omega}$, and T-periodic in t; $H_u(x) \not\equiv 0$ is Hölder continuous and nonnegative.

In the following, we first study the well-posedness for system (1). Let $\mathbb{X} := C(\bar{\Omega}, \mathbb{R}^3)$ be the Banach space with the supremum norm $\|\cdot\|_{\mathbb{X}}$. Define $\mathbb{X}^+ := C(\bar{\Omega}, \mathbb{R}^3_+)$. Then $(\mathbb{X}, \mathbb{X}^+)$ is a strongly ordered Banach space. From the last two equations V_u and V_i in system (1), we have

$$
\begin{aligned}
\frac{\partial (V_u(t,x) + V_i(t,x))}{\partial t} &= \nabla \cdot (\delta_2(t,x)\nabla(V_u(t,x) + V_i(t,x))) \\
&\quad + \beta(t,x)(V_u(t,x) + V_i(t,x)) \\
&\quad - \mu_1(t,x)(V_u(t,x) + V_i(t,x)) \\
&\quad - \mu_2(t,x)(V_u(t,x) + V_i(t,x))^2.
\end{aligned}
$$

Let $V(t,x) = V_u(t,x) + V_i(t,x)$ be the total density of the vector population. It then follows that

$$
\begin{aligned}
\tfrac{\partial V(t,x)}{\partial t} &= \nabla \cdot (\delta_2(t,x)\nabla V(t,x)) + \beta(t,x)V(t,x) - \mu_1(t,x)V(t,x) \\
&\quad - \mu_2(t,x)V^2(t,x), t > 0, x \in \Omega, \\
\tfrac{\partial V}{\partial \nu} &= 0, t > 0, x \in \partial\Omega.
\end{aligned}
\tag{2}
$$

To ensure that there is a globally stable positive periodic state for the mosquito population, we assume that

(A2) $\int_0^T \int_\Omega \beta(t,x)\mathrm{d}x\mathrm{d}t > \int_0^T \int_\Omega \mu_1(t,x)\mathrm{d}x\mathrm{d}t$.

Let μ be the principle eigenvalue of the periodic parabolic problem

$$\begin{aligned}\frac{\partial V(t,x)}{\partial t} &= \nabla\cdot(\delta_2(t,x)\nabla V(t,x)) + \beta(t,x)V(t,x)\\&\quad - \mu_1(t,x)V(t,x) + \mu V(t,x),\ t>0,\ x\in\Omega,\\ \frac{\partial V}{\partial \nu} &= 0,\ t>0,\ x\in\partial\Omega.\end{aligned}$$

In view of (A2), it follows from Hess (1991, Section 7.1) that $\mu < 0$. By a standard convergence result on the periodic parabolic logistic equations (see, e.g., Hess 1991, Theorem 28.1 or Zhao 2017b, Theorem 3.1.5), we see that system (2) has a globally stable positive T-periodic solution $V^*(t,x)$, that is, $\lim_{t\to\infty}(V(t,x) - V^*(t,x)) = 0$ uniformly for all $x \in \Omega$, for any $V(0,\cdot) \geq 0$ but $V(0,\cdot) \not\equiv 0$. Clearly, system (1) is equivalent to the following one:

$$\begin{aligned}\frac{\partial H_i(t,x)}{\partial t} &= \nabla\cdot(\delta_1(t,x)\nabla H_i(t,x)) - \lambda(t,x)H_i(t,x)\\&\quad + \sigma_1(t,x)H_u(x)V_i(t,x),\ t>0,\ x\in\Omega,\\ \frac{\partial V_i(t,x)}{\partial t} &= \nabla\cdot(\delta_2(t,x)\nabla V_i(t,x)) + \sigma_2(t,x)(V(t,x)\\&\quad - V_i(t,x))H_i(t,x) - \mu_1(t,x)V_i(t,x)\\&\quad - \mu_2(t,x)V(t,x)V_i(t,x),\ t>0,\ x\in\Omega,\\ \frac{\partial V(t,x)}{\partial t} &= \nabla\cdot(\delta_2(t,x)\nabla V(t,x)) + \beta(t,x)V(t,x) - \mu_1(t,x)V(t,x)\\&\quad - \mu_2(t,x)V^2(t,x),\ t>0,\ x\in\Omega,\\ \frac{\partial H_i}{\partial \nu} &= \frac{\partial V_i}{\partial \nu} = \frac{\partial V}{\partial \nu} = 0,\ t>0,\ x\in\partial\Omega,\end{aligned}\tag{3}$$

with its phase space being

$$X := \{\varphi = (\varphi_1,\varphi_2,\varphi_3) \in \mathbb{X}^+ : \varphi_1(x) \geq 0,\ 0 \leq \varphi_2(x) \leq \varphi_3(x),\ \forall x \in \bar{\Omega}\}.$$

Let $\mathbb{Y} := C(\bar{\Omega},\mathbb{R})$ and $\mathbb{Y}^+ := C(\bar{\Omega},\mathbb{R}_+)$. Let $T_1(t,s), T_2(t,s) : \mathbb{Y} \to \mathbb{Y}$, $t \geq s$, be the linear evolution operators associated with

$$\frac{\partial u_1}{\partial t} = \nabla\cdot(\delta_1(t,x)\nabla u_1(t,x)) - \lambda(t,x)u_1(t,x),\ t>0,\ x\in\Omega,$$

and

$$\frac{\partial u_2}{\partial t} = \nabla\cdot(\delta_2(t,x)\nabla u_2(t,x)) - \mu_1(t,x)u_2(t,x),\ t>0,\ x\in\Omega,$$

subject to the Neumann boundary condition, respectively. Since $\lambda(t,\cdot)$ and $\mu_1(t,\cdot)$ are T-periodic in t, it follows from Daners and Medina (1992, Lemma 6.1) that $T_i(t+T, s+T) = T_i(t,s)$ for any $(t,s) \in \mathbb{R}^2$ with $t \geq s$, $i = 1, 2$. Moreover, for any $(t,s) \in \mathbb{R}^2$ with $t > s$, $T_i(t,s)$, $i = 1, 2$, is compact and strongly positive (see, e.g., Hess 1991, Chapter II and Smith 1995, Theorems 7.3.1 and 7.4.1). Let $T(t,s) := \text{diag}\{T_1(t,s), T_2(t,s), T_2(t,s)\} : \mathbb{X} \to \mathbb{X}$, $\forall t \geq s$, and $\mathscr{A}(t) := \text{diag}\{A_1(t), A_2(t), A_2(t)\}$, where $A_1(t)$ and $A_2(t)$ are defined by,

$$\begin{aligned} D(A_i(t)) &= \left\{\varphi \in C^2(\bar{\Omega}) : \frac{\partial \varphi}{\partial \nu} = 0 \text{ on } \partial\Omega\right\},\ i = 1, 2, \\ A_1(t)\varphi &= \nabla \cdot (\delta_1(t,x)\nabla\varphi(t,x)) - \lambda(t,x)\varphi(t,x),\ \forall \varphi \in D(A_1(t)), \\ A_2(t)\varphi &= \nabla \cdot (\delta_2(t,x)\nabla\varphi(t,x)) - \mu_1(t,x)\varphi(t,x),\ \forall \varphi \in D(A_2(t)). \end{aligned}$$

Define $F = (F_1, F_2, F_3) : [0, +\infty) \times X \to \mathbb{X}$ by

$$\begin{aligned} F_1(t,\varphi) &= \sigma_1(t,\cdot)H_u(\cdot)\varphi_2(\cdot), \\ F_2(t,\varphi) &= \sigma_2(t,\cdot)(\varphi_3(\cdot) - \varphi_2(\cdot))\varphi_1(\cdot) - \mu_2(t,\cdot)\varphi_3(\cdot)\varphi_2(\cdot), \\ F_3(t,\varphi) &= \beta(t,\cdot)\varphi_3(\cdot) - \mu_2(t,\cdot)\varphi_3^2(\cdot), \end{aligned}$$

for all $t \geq 0$ and $\varphi = (\varphi_1, \varphi_2, \varphi_3) \in X$. Then system (3) can be written as the following abstract differential equation:

$$\begin{cases} \frac{\mathrm{d}u}{\mathrm{d}t} = \mathscr{A}(t)u + F(t,u), t > 0, \\ u_{(0)} = \varphi \in X. \end{cases} \tag{4}$$

Lemma 1 *Let (A1)–(A2) hold. For any $\varphi \in X$, system (3) has a unique nonnegative solution $u(t,\cdot,\varphi)$ with $u(0,\cdot,\varphi) = \varphi$ such that $u(t,\cdot,\varphi) \in X$ for all $t \in [0, +\infty)$, and solutions are ultimately bounded and uniformly bounded.*

Proof By the abstract setting in Martin and Smith (1990), we consider the integral form of system (4):

$$\begin{aligned} u(t,\varphi) &= T(t,0)\varphi + \textstyle\int_0^t T(t,s)F(s,u(s))\mathrm{d}s, t > 0, \\ u_{(0)} &= \varphi \in X. \end{aligned}$$

From the expression of F, we see that F is locally Lipschitz continuous. For any $(t,\varphi) \in \mathbb{R}_+ \times X$ and $k > 0$, we have

$$\begin{aligned} &\varphi(x) + kF(t,\varphi)(x) \\ &= \begin{pmatrix} \varphi_1(x) + k\sigma_1(t,x)H_u(x)\varphi_2(x) \\ \varphi_2(x) + k[\sigma_2(t,x)(\varphi_3(x) - \varphi_2(x))\varphi_1(x) - \mu_2(t,x)\varphi_3(x)\varphi_2(x)] \\ \varphi_3(x) + k[\beta(t,x)\varphi_3(x) - \mu_2(t,x)\varphi_3^2(x)] \end{pmatrix} \end{aligned}$$

$$\geq \begin{pmatrix} \varphi_1(x) \\ \varphi_2(x)[1 - k(\hat{\sigma}_2\varphi_1(x) + \hat{\mu}_2\varphi_3(x))] \\ \varphi_3(x)(1 - k\hat{\mu}_2\varphi_3(x)) \end{pmatrix},$$

where $\hat{\mu}_2 = \max_{t\in[0,T],x\in\bar{\Omega}} \mu_2(t,x)$, $\hat{\sigma}_2 = \max_{t\in[0,T],x\in\bar{\Omega}} \sigma_2(t,x)$, and

$$\begin{aligned}\varphi_3(x) - [\varphi_2(x) + kF_2(t,\varphi)(x)] &= \varphi_3(x) - \varphi_2(x) - k\sigma_2(t,x)(\varphi_3(x) - \varphi_2(x))\varphi_1(x) \\ &\quad + k\mu_2(x)\varphi_3(x)\varphi_2(x) \\ &\geq (\varphi_3(x) - \varphi_2(x))(1 - k\hat{\sigma}_2\varphi_1(x)).\end{aligned}$$

Thus, for sufficiently small $k > 0$, we have $\varphi + kF(t,\varphi) \in X$, and hence,

$$\lim_{k\to 0^+} \frac{1}{k}\text{dist}(\varphi + kF(t,\varphi), X) = 0,\ \forall (t,\varphi) \in \mathbb{R}_+ \times X.$$

Since $T_1(t,s)$ and $T_2(t,s)$ are positive for $t \geq s$, we easily see that $T(t,s) : X \to X$ for all $t \geq s \geq 0$. By Martin and Smith (1990, Corollary 4) with $K = X$ and $S(t,s) = T(t,s)$, it then follows that for any $\varphi \in X$, system (3) has a unique non-continuable mild solution $u(t,\cdot,\varphi)$ with $u_0 = \varphi$ on its maximal existence interval $t \in [0,t_\varphi)$, where $t_\varphi \leq +\infty$, and $u(t,\cdot,\varphi) \in X$ for all $t \in [0,t_\varphi)$. Moreover, by the analyticity of $T(t,s)$ with respect to $(t,s) \in \mathbb{R}^2$, $t > s$, $u(t,\cdot,\varphi)$ is a classical solution of system (3) for $t > 0$.

Since system (2) admits a globally stable positive T-periodic solution $V^*(t,x)$, it follows that $u_3(t,\cdot,\varphi) = V(t,\cdot)$ is bounded on $[0,t_\varphi)$, and hence, $u_2(t,\cdot,\varphi) = V_i(t,\cdot)$ is also bounded on $[0,t_\varphi)$. Then there exists a constant $B > 0$ such that the first equation H_i of system (3) is dominated by the following one:

$$\begin{aligned}&\tfrac{\partial w(t,x)}{\partial t} = \nabla\cdot(\delta_1(t,x)\nabla w(t,x)) - \lambda(t,x)w(t,x) + B, t > 0, x \in \Omega, \\ &\tfrac{\partial w}{\partial \nu} = 0, t > 0, x \in \partial\Omega.\end{aligned}$$

By Zhang et al. (2015, Lemma 2.1) and the comparison principle, $u_1(t,\cdot,\varphi) = H_i(t,\cdot)$ is bounded on $[0,t_\varphi)$. Thus, the solution $u(t,\cdot,\varphi)$ is bounded on $[0,t_\varphi)$, and hence, $t_\varphi = +\infty$ for any $\varphi \in X$. It then follows from the comparison argument that solutions of system (3) with initial data in X exist globally on $[0,+\infty)$ and are also ultimately bounded.

It is easy to see that there exists a positive vector $\zeta = (\zeta_1,\zeta_2,\zeta_3) := (\frac{\hat{\sigma}_1\hat{H}_u(\hat{\beta}-\bar{\mu}_1)}{\bar{\mu}_2\bar{\lambda}}, \frac{\hat{\beta}-\bar{\mu}_1}{\bar{\mu}_2}, \frac{\hat{\beta}-\bar{\mu}_1}{\bar{\mu}_2})$ such that

$$\begin{aligned}&\sigma_1(t,x)H_u(x)\zeta_2 - \lambda(t,x)\zeta_1 \leq 0, -\mu_1(t,x)\zeta_2 - \mu_2(t,x)\zeta_2\zeta_3 \leq 0, \\ &\quad(\beta(t,x) - \mu_1(t,x))\zeta_3 - \mu_2(t,x)\zeta_3^2 \leq 0,\end{aligned}$$

where $\hat{\sigma}_1 = \max_{t\in[0,T],x\in\bar{\Omega}} \sigma_1(t,x)$, $\hat{H}_u = \max_{x\in\bar{\Omega}} H_u(x)$, $\hat{\beta} = \max_{t\in[0,T],x\in\bar{\Omega}} \beta(t,x)$, $\bar{\mu}_i = \min_{t\in[0,T],x\in\bar{\Omega}} \mu_i(t,x)$ $(i = 1,2)$, and $\bar{\lambda} = \min_{t\in[0,T],x\in\bar{\Omega}} \lambda(t,x)$. Thus, for any $m \geq 1$, $m\zeta$ is an upper solution of systems (3). This implies that solutions of system (3) are uniformly bounded. □

3 Threshold Dynamics

In this section, we first introduce the basic reproduction ratio R_0 for system (3) and then establish a threshold type result on its global dynamics in terms of R_0.

Let $\mathbb{E} := C(\bar{\Omega}, \mathbb{R}^2)$, $\mathbb{E}^+ := C(\bar{\Omega}, \mathbb{R}^2_+)$, and $C_T(\mathbb{R}, \mathbb{E})$ be the Banach space consisting of T-periodic and continuous functions from $\mathbb{R}$ to $\mathbb{E}$, where $\|\phi\|_{C_T(\mathbb{R},\mathbb{E})} := \max_{\theta \in [0,T]} \|\phi(\theta)\|_{\mathbb{E}}$ for any $\phi \in C_T(\mathbb{R}, \mathbb{E})$. Letting $H_i = V_i = 0$ in system (3), we obtain system (2). Thus, linearizing system (3) at the disease-free periodic solution $(0, 0, V^*(t, x))$, we consider the following system of infectious compartments:

$$\begin{aligned} \frac{\partial H_i(t,x)}{\partial t} &= \nabla \cdot (\delta_1(t,x) \nabla H_i(t,x)) - \lambda(t,x) H_i(t,x) \\ &\quad + \sigma_1(t,x) H_u(x) V_i(t,x), \, t > 0, x \in \Omega, \\ \frac{\partial V_i(t,x)}{\partial t} &= \nabla \cdot (\delta_2(t,x) \nabla V_i(t,x)) + \sigma_2(t,x) V^*(t,x) H_i(t,x) \\ &\quad - (\mu_1(t,x) + \mu_2(t,x) V^*(t,x)) V_i(t,x), \, t > 0, x \in \Omega, \\ \frac{\partial H_i}{\partial \nu} &= \frac{\partial V_i}{\partial \nu} = 0, \, t > 0, x \in \partial \Omega. \end{aligned} \tag{5}$$

Define the operator $F(t) : \mathbb{E} \to \mathbb{E}$ by

$$F(t) \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} \sigma_1(t,\cdot) H_u(\cdot) \phi_2(\cdot) \\ \sigma_2(t,\cdot) V^*(t,\cdot) \phi_1(\cdot) \end{pmatrix}, \ \forall t \in \mathbb{R}, \phi = (\phi_1, \phi_2) \in \mathbb{E}.$$

Let $-W(t)v = \nabla \cdot (\delta(t,\cdot) \nabla v) - \mathscr{W}(t) v$, where $\delta(t,\cdot) = \mathrm{diag}(\delta_1(t,\cdot), \delta_2(t,\cdot))$ and

$$-[\mathscr{W}(t)](x) = \begin{pmatrix} -\lambda(t,x) & 0 \\ 0 & -(\mu_1(t,x) + \mu_2(t,x) V^*(t,x)) \end{pmatrix}, \ \forall t \in \mathbb{R}, x \in \bar{\Omega}.$$

Then system (5) can be written as

$$\frac{\mathrm{d}v}{\mathrm{d}t} = F(t)v - W(t)v, \, t \geq 0.$$

Let $T_3(t, s)$, $t \geq s$, be the evolution operator on $\mathbb{Y}$ associated with

$$\begin{aligned} \frac{\partial u(t,x)}{\partial t} &= \nabla \cdot (\delta_2(t,x) \nabla u(t,x)) - (\mu_1(t,x) \\ &\quad + \mu_2(t,x) V^*(t,x)) u(t,x), \, t > 0, x \in \Omega, \\ \frac{\partial u}{\partial \nu} &= 0, \, t > 0, x \in \partial \Omega. \end{aligned}$$

Thus, $\Phi(t, s) := \mathrm{diag}(\mathrm{T}_1(\mathrm{t, s}), \mathrm{T}_3(\mathrm{t, s}))$, $t \geq s$, is the evolution family on $\mathbb{E}$ associated with the linear system

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -W(t)v.$$

The exponential growth bound of $\Phi(t, s)$ is defined by

$$\bar{\omega}(\Phi) = \inf\{\tilde{\omega} : \exists M \geq 1 \text{ such that } \|\Phi(t+s, s)\| \leq Me^{\tilde{\omega}t}, \ \forall s \in \mathbb{R}, \ t \geq 0\}.$$

By the Krein–Rutman Theorem (see, e.g., Hess 1991, Theorem 7.2 and Lemma 14.2), we have

$$0 < r(\Phi(T, 0)) = \max\{r(T_1(T, 0)), r(T_3(T, 0))\} < 1,$$

where $r(\Phi(T, 0))$ is the spectral radius of $\Phi(T, 0)$. Thus, Thieme (2009, Proposition 5.6) with $s = 0$ implies that $\bar{\omega}(\Phi) < 0$. Note that $\Phi(t, s)$ is a positive operator in the sense that $\Phi(t, s)\mathbb{E}^+ \subseteq \mathbb{E}^+$ for all $t \geq s$. Therefore, $F(t)$ and $\Phi(t, s)$ satisfy the following assumptions:

(H1) For each $t \geq 0$, $F(t)$ is a positive operator on $\mathbb{E}$.
(H2) For any $t \geq s$, $\Phi(t, s)$ is a positive operator on $\mathbb{E}$, and $\bar{\omega}(\Phi) < 0$.

Following Liang et al. (2019) and Zhao (2017a), we define two linear operators on $C_T(\mathbb{R}, \mathbb{E})$ by

$$[Lv](t) := \int_0^{+\infty} \Phi(t, t-s)F(t-s)v(t-s)\mathrm{d}s, \ \forall t \in \mathbb{R}, \ v \in C_T(\mathbb{R}, \mathbb{E}),$$

and

$$[\mathscr{L}v](t) := F(t)\left(\int_0^{+\infty} \Phi(t, t-s)v(t-s)\mathrm{d}s\right), \ \forall t \in \mathbb{R}, \ v \in C_T(\mathbb{R}, \mathbb{E}).$$

Let A and B be two bounded linear operators on $C_T(\mathbb{R}, \mathbb{E})$ given by

$$[Av](t) := \int_0^{+\infty} \Phi(t, t-s)v(t-s)\mathrm{d}s, \ [Bv](t) = F(t)v, \ \forall t \in \mathbb{R}, v \in C_T(\mathbb{R}, \mathbb{E}).$$

We then have $L = A \circ B$ and $\mathscr{L} = B \circ A$, and hence, L and $\mathscr{L}$ have the same spectral radius. Thus, we define the basic reproduction ratio as $R_0 := r(L) = r(\mathscr{L})$, where $r(L)$ and $r(\mathscr{L})$ are the spectral radii of L and $\mathscr{L}$, respectively.

For any given $t \geq 0$, let $P(t)$ be the solution map of system (5) on $\mathbb{E}$, that is, $P(t)\phi = v(t, \phi)$, where $v(t, \phi)(x) = v(t, x, \phi)$, $\forall x \in \bar{\Omega}$, and $v(t, x, \phi)$ is the unique solution of system (5) with $v(0, x, \phi) = \phi(x)$, $\forall x \in \bar{\Omega}$. Then, $P := P(T)$ is the Poincaré map associated with system (5). Let $r(P)$ be the spectral radius of P. By Liang et al. (2019, Theorem 3.7) with $\tau = 0$, we have the following result.

Lemma 2 $R_0 - 1$ *has the same sign as* $r(P) - 1$.

Note that for any $V(0,\cdot) \geq 0$ but $V(0,\cdot) \not\equiv 0$, there holds $\lim_{t\to\infty}(V(t,x) - V^*(t,x)) = 0$ uniformly for all $x \in \bar{\Omega}$. We then consider the following limiting system:

$$\begin{aligned}
\frac{\partial H_i(t,x)}{\partial t} &= \nabla \cdot (\delta_1(t,x)\nabla H_i(t,x)) - \lambda(t,x)H_i(t,x) \\
&\quad + \sigma_1(t,x)H_u(x)V_i(t,x),\ t > 0, x \in \Omega, \\
\frac{\partial V_i(t,x)}{\partial t} &= \nabla \cdot (\delta_2(t,x)\nabla V_i(t,x)) + \sigma_2(t,x)(V^*(t,x) - V_i(t,x))H_i(t,x) \\
&\quad - \mu_1(t,x)V_i(t,x) - \mu_2(t,x)V^*(t,x)V_i(t,x),\ t > 0, x \in \Omega, \\
\frac{\partial H_i}{\partial \nu} &= \frac{\partial V_i}{\partial \nu} = 0,\ t > 0, x \in \partial\Omega.
\end{aligned} \tag{6}$$

For each $t \geq 0$, we define

$$E(t) := \{(\varphi_1, \varphi_2) \in \mathbb{E}^+ : \varphi_1(x) \geq 0, 0 \leq \varphi_2(x) \leq V^*(t,x), \forall x \in \bar{\Omega}\}.$$

Then, we have the following result for system (6).

Lemma 3 *Let (A1)–(A2) hold. For any $\phi \in E(0)$, system (6) has a unique solution $v(t,\cdot,\phi)$ with $v(0,\cdot,\phi) = \phi$ such that $v(t,\cdot,\phi) = (v_1(t,\cdot,\phi), v_2(t,\cdot,\phi)) \in E(t)$ for all $t \geq 0$, and solutions are ultimately bounded and uniformly bounded. Moreover, system (6) generates a T-periodic semiflow $Q(t) := v(t,\cdot) : E(0) \to E(t)$.*

Proof From system (6) with the initial data ϕ, we have

$$\begin{aligned}
v_1(t,\cdot,\phi) &= T_1(t,0)\phi_1(\cdot) + \int_0^t T_1(t,s)[\sigma_1(s,\cdot)H_u(\cdot)v_2(s,\cdot)]\mathrm{d}s, \\
v_2(t,\cdot,\phi) &= T_3(t,0)\phi_2(\cdot) + \int_0^t T_3(t,s)[\sigma_2(s,\cdot)(V^*(s,\cdot) - v_2(s,\cdot))v_1(s,\cdot)]\mathrm{d}s.
\end{aligned}$$

Define $\tilde{F} = (\tilde{F}_1, \tilde{F}_2) : [0,+\infty) \times \mathbb{E} \to \mathbb{E}$ by

$$\begin{aligned}
&\tilde{F}_1(t,\phi) := \sigma_1(t,\cdot)H_u(\cdot)\phi_2(\cdot), \\
&\tilde{F}_2(t,\phi) := \sigma_2(t,\cdot)(V^*(t,\cdot) - \phi_2(\cdot)))\phi_1(\cdot), \forall t \geq 0, \phi = (\phi_1,\phi_2) \in \mathbb{E}.
\end{aligned}$$

Then, system (6) can be written as an integral equation

$$v(t,\phi) = \Phi(t,0)\phi + \int_0^t \Phi(t,s)\tilde{F}(s,v(s))\mathrm{d}s, \forall t \geq 0, \phi \in \mathbb{E}.$$

We first show that $\tilde{F}$ is quasi-monotone on $E := \{(t,\phi) \in [0,+\infty) \times \mathbb{E}^+ :\ \phi \in E(t)\}$ in the sense that

$$\lim_{k\to 0^+} \frac{1}{k}\mathrm{dist}((\psi - \phi) + k[\tilde{F}(t,\psi) - \tilde{F}(t,\phi)], \mathbb{E}^+) = 0, \tag{7}$$

for all $(t, \phi), (t, \psi) \in E$ with $\phi \leq \psi$. Indeed, for any given $(t, \phi), (t, \psi) \in E$ with $\phi \leq \psi$, we have

$$\begin{aligned}
&\psi - \phi + k[\tilde{F}(t, \psi) - \tilde{F}(t, \phi)] \\
&= \begin{pmatrix} \psi_1(\cdot) - \phi_1(\cdot) + k[\sigma_1(t, \cdot)H_u(\cdot)\psi_2(\cdot) - \sigma_1(t, \cdot)H_u(\cdot)\phi_2(\cdot)] \\ \psi_2(\cdot) - \phi_2(\cdot) + k[\sigma_2(t, \cdot)(V^*(t, \cdot) - \psi_2(\cdot))\psi_1(\cdot) \\ \quad - \sigma_2(t, \cdot)(V^*(t, \cdot) - \phi_2(\cdot))\phi_1(\cdot)] \end{pmatrix} \\
&\geq \begin{pmatrix} \psi_1(\cdot) - \phi_1(\cdot) + k\bar{\sigma}_1 \bar{H}_u(\psi_2(\cdot) - \phi_2(\cdot)) \\ \psi_2(\cdot) - \phi_2(\cdot) + k\bar{\sigma}_2 V^*(t, \cdot)(\psi_1(\cdot) - \phi_1(\cdot)) - k\bar{\sigma}_2(\psi_1(\cdot)\psi_2(\cdot) - \phi_1(\cdot)\phi_2(\cdot)) \end{pmatrix} \\
&= \begin{pmatrix} \psi_1(\cdot) - \phi_1(\cdot) + k\bar{\sigma}_1 \bar{H}_u(\psi_2(\cdot) - \phi_2(\cdot)) \\ (1 - k\bar{\sigma}_2\phi_1(\cdot))(\psi_2(\cdot) - \phi_2(\cdot)) + k\bar{\sigma}_2(V^*(t, \cdot) - \psi_2(\cdot))(\psi_1(\cdot) - \phi_1(\cdot)) \end{pmatrix},
\end{aligned}$$

where $\bar{\sigma}_i = \min_{t \in [0,T], x \in \bar{\Omega}} \sigma_i(t, x)$, $i = 1, 2$, $\bar{H}_u = \min_{x \in \bar{\Omega}} H_u(x)$.

Thus, $\psi - \phi + k[\tilde{F}(t, \psi) - \tilde{F}(t, \phi)] \in \mathbb{E}^+$ for all sufficiently small $k > 0$, and hence, (7) holds true. Letting $v^-(t) = (0, 0)$, $v^+(t) = (+\infty, V^*(t, \cdot))$, $S^+ = S^- = S = \Phi$, and $B^- = B^+ = \tilde{F}$, we can easily verify assumptions (C1)–(C6) in Martin and Smith (1990). It then follows from Martin and Smith (1990, Corallary 5) that for any $\phi \in E(0)$, system (6) admits a unique solution $v(t, \cdot, \phi)$ with $v(0, \cdot, \phi) = \phi$ such that $v(t, \cdot, \phi) \in E(t)$ for all t in its maximal interval of existence $[0, t_\phi)$.

By the arguments similar to those for Lemma 1, it follows that the solution of system (6) with initial data $\phi \in E(0)$ exists globally on $[0, +\infty)$, and solutions are ultimately bounded and uniformly bounded. For any given $t \geq 0$, we define an operator $Q(t) : E(0) \to E(t)$ by $Q(t)(\phi)(x) = v(t, x, \phi)$, $\forall \phi \in E(0)$, $x \in \bar{\Omega}$. It then follows that $Q(t) : E(0) \to E(t)$ is a T-periodic semiflow (see, e.g., Zhao 2017b), and $Q := Q(T) : E(0) \to E(T) = E(0)$ is the Poincaré map associated with system (6). □

Lemma 4 *Let (A1)–(A2) hold. For any ϕ and ψ in $E(0)$ with $\phi > \psi$ (that is, $\phi \geq \psi$, but $\phi \not\equiv \psi$), the solutions $\bar{v}(t, \cdot, \phi)$ and $v(t, \cdot, \psi)$ of system (6) with $\bar{v}_0(0, \cdot, \phi) = \phi$ and $v_0(0, \cdot, \psi) = \psi$, respectively, satisfy $\bar{v}(t, \cdot, \phi) \gg v(t, \cdot, \psi)$ for all $t > 0$. That is, the map $Q(t) : E(0) \to E(t)$ is strongly monotone for each $t > 0$.*

Proof The comparison theorem for cooperative parabolic systems implies that $\bar{v}_i(t, \cdot, \phi) \geq v_i(t, \cdot, \psi)$ for all $t \geq 0$, $i = 1, 2$. Let $\phi, \psi \in E(0)$ satisfy $\phi > \psi$. Denote $\bar{v}(t, \cdot) = \bar{v}(t, \cdot, \phi) = (\bar{v}_1(t, \cdot), \bar{v}_2(t, \cdot))$ and $v(t, \cdot) = v(t, \cdot, \psi) = (v_1(t, \cdot), v_2(t, \cdot))$. Without loss of generality, we assume that $\phi_1(\cdot) > \psi_1(\cdot)$. Clearly, we have

$$\begin{aligned}
\frac{\partial(\bar{v}_1(t, x) - v_1(t, x))}{\partial t} &= \nabla \cdot (\delta_1(t, x)\nabla(\bar{v}_1(t, x) - v_1(t, x))) \\
&\quad - \lambda(t, x)(\bar{v}_1(t, x) - v_1(t, x)) \\
&\quad + \sigma_1(t, x)H_u(x)(\bar{v}_2(t, x) - v_2(t, x)) \\
&\geq \nabla \cdot (\delta_1(t, x)\nabla(\bar{v}_1(t, x) - v_1(t, x))) \\
&\quad - \lambda(t, x)(\bar{v}_1(t, x) - v_1(t, x)), \\
\frac{\partial(\bar{v}_1 - v_1)}{\partial \nu} &= 0, \ t > 0, \ x \in \partial\Omega,
\end{aligned}$$

$$\bar{v}_1(0,\cdot) - v_1(0,\cdot) = \phi_1(\cdot) - \psi_1(\cdot) > 0.$$

By the standard parabolic comparison theorem and maximal principle, it follows that $\bar{v}_1(t,x) > v_1(t,x)$ for all $t > 0$ and $x \in \bar{\Omega}$, i.e., $\bar{v}_1(t,\cdot) \gg v_1(t,\cdot)$ for each $t > 0$. By an argument similar to that for $\bar{v}_1$ and v_1, we have $\bar{v}_2(t,\cdot) \gg v_2(t,\cdot)$ for each $t > 0$ provided that $\phi_2(\cdot) > \psi_2(\cdot)$. It suffices to consider the case where $\phi_2(\cdot) \equiv \psi_2(\cdot)$. For such a case, we have the following claim.

Claim. $\bar{v}_2(t,\cdot) > v_2(t,\cdot)$ for all $t > 0$.

Indeed, assume, by contradiction, that $\bar{v}_2(t_0,\cdot) = v_2(t_0,\cdot)$ for some $t_0 > 0$. Then, the maximum principle implies that $\bar{v}_2(t,\cdot) = v_2(t,\cdot)$, $\forall t \in [0,t_0]$, and hence, $\frac{\partial \bar{v}_2(t,x)}{\partial t} = \frac{\partial v_2(t,x)}{\partial t}$ for all $t \in [0,t_0]$ and $x \in \bar{\Omega}$. It then follows that

$$\sigma_2(t,\cdot)(V^*(t,\cdot) - v_2(t,\cdot))(\bar{v}_1(t,\cdot) - v_1(t,\cdot)) = 0, \forall t \in [0,t_0].$$

Since $\bar{v}_1(t,\cdot) \gg v_1(t,\cdot)$, $\forall t > 0$, we have $V^*(t,\cdot) = v_2(t,\cdot)$, $\forall t \in [0,t_0]$, and hence,

$$\begin{aligned}\frac{\partial V^*(t,x)}{\partial t} = \frac{\partial v_2(t,x)}{\partial t} &= \nabla \cdot (\delta_2(t,x)\nabla v_2(t,x)) - (\mu_1(t,x) \\ &\quad + \mu_2(t,x)V^*(t,x))v_2(t,x),\end{aligned}$$

for all $t \in [0,t_0]$, which contradicts the fact that

$$\begin{aligned}\frac{\partial V^*(t,x)}{\partial t} &= \nabla \cdot (\delta_2(t,x)\nabla V^*(t,x)) + \beta(t,x)V^*(t,x) - (\mu_1(t,x) \\ &\quad + \mu_2(t,x)V^*(t,x))V^*(t,x).\end{aligned}$$

This proves the claim above.

Let $g_1(t,x,\xi) := \nabla \cdot (\delta_2(t,x)\nabla \xi) + \sigma_2(t,x)(V^*(t,x) - \xi)v_1(t,x) - (\mu_1(t,x) + \mu_2(t,x) \times V^*(t,x))\xi$, $x \in \bar{\Omega}$. Since

$$\begin{aligned}\frac{\partial \bar{v}_2(t,x)}{\partial t} &= \nabla \cdot (\delta_2(t,x)\nabla \bar{v}_2(t,x)) + \sigma_2(t,x)(V^*(t,x) - \bar{v}_2(t,x))\bar{v}_1(t,x) \\ &\quad - (\mu_1(t,x) + \mu_2(t,x)V^*(t,x))\bar{v}_2(t,x) \\ &\geq \nabla \cdot (\delta_2(t,x)\nabla \bar{v}_2(t,x)) + \sigma_2(t,x)(V^*(t,x) - \bar{v}_2(t,x))v_1(t,x) \\ &\quad - (\mu_1(t,x) + \mu_2(t,x)V^*(t,x))\bar{v}_2(t,x) \\ &= g_1(t,x,\bar{v}_2(t,x)),\end{aligned}$$

we have $\frac{\partial \bar{v}_2(t,x)}{\partial t} - g_1(t,x,\bar{v}_2(t,x)) \geq \frac{\partial v_2(t,x)}{\partial t} - g_1(t,x,v_2(t,x))$, $\forall t > 0$, $x \in \Omega$, with $\frac{\partial \bar{v}_2(t,x)}{\partial t} = \frac{\partial v_2(t,x)}{\partial t} = 0$, $x \in \partial\Omega$. For any given $t_1 > 0$, the above claim implies that $\bar{v}_2(t_1,\cdot) > v_2(t_1,\cdot)$. It then follows from the parabolic maximum principle that $\bar{v}_2(t,\cdot) \gg v_2(t,\cdot)$ for all $t > t_1$. Since $t_1 > 0$ is arbitrary, we have $\bar{v}_2(t,\cdot) \gg v_2(t,\cdot)$ for all $t > 0$. Thus, $\bar{v}(t,\cdot,\phi) \gg v(t,\cdot,\psi)$ for all $t > 0$. □

By the continuity and differentiability of solutions with respect to the initial data, it is easy to see that Q is differentiable at zero and the Frechét derivative $DQ(0) = P$.

In the following, we establish a threshold type result on the global dynamics of system (6) in terms of R_0.

Theorem 1 *Assume that (A1)–(A2) hold. The following statements are valid:*

(i) If $R_0 \leq 1$, then $(0, 0)$ is globally asymptotically stable for system (6) in $E(0)$.
(ii) If $R_0 > 1$, then system (6) admits a unique positive T-periodic solution $(H_i^*(t,x), V_i^*(t,x))$, and it is globally asymptotically stable for system (6) in $E(0) \setminus \{(0,0)\}$.

Proof For any given $\phi \in E(0)$ and $\alpha \in [0, 1]$, let $v(t, x, \phi)$ and $v(t, x, \alpha\phi)$ be the solutions of system (6) with $v(0, x, \phi) = \phi(x)$ and $v(0, x, \alpha\phi) = \alpha\phi(x)$, $x \in \bar{\Omega}$, respectively. Define $u(t, x) := \alpha v(t, x, \phi) = (u_1(t, x), u_2(t, x)))$ and $w(t, x) := v(t, x, \alpha\phi) = (w_1(t, x), w_2(t, x))$, we then have

$$
\begin{aligned}
\frac{\partial u_1(t,x)}{\partial t} &= \alpha\frac{\partial v_1(t,x)}{\partial t} = \nabla \cdot (\delta_1(t,x)\nabla(\alpha v_1(t,x))) - \lambda(t,x)(\alpha v_1(t,x)) \\
&\quad + \sigma_1(t,x)H_u(x)(\alpha v_2(t,x)), \\
\frac{\partial u_2(t,x)}{\partial t} &= \alpha\frac{\partial v_2(t,x)}{\partial t} = \nabla \cdot (\delta_2(t,x)\nabla(\alpha v_2(t,x))) + \sigma_2(t,x)(V^*(t,x) \\
&\quad - v_2(t,x))(\alpha v_1(t,x)) - (\mu_1(t,x) + \mu_2(t,x)V^*(t,x))(\alpha v_2(t,x)) \\
&\leq \nabla \cdot (\delta_2(t,x)\nabla(\alpha v_2(t,x))) + \sigma_2(t,x)(V^*(t,x) - \alpha v_2(t,x))(\alpha v_1(t,x)) \\
&\quad - (\mu_1(t,x) + \mu_2(t,x)V^*(t,x))(\alpha v_2(t,x)), t > 0, x \in \Omega.
\end{aligned}
$$

Thus, $u(t, x)$ is a lower solution of system (6) with $u(0, x) = \alpha v(0, x, \phi) = \alpha\phi(x)$, $x \in \bar{\Omega}$. It then follows that $\alpha v(t, x, \phi) \leq v(t, x, \alpha\phi)$ for all $t \geq 0, x \in \bar{\Omega}$. This shows that the solution map $Q(t) : E(0) \to E(t)$ is subhomogeneous. Moreover, we have the following claim.

Claim. For each $t > 0$, $Q(t) : E(0) \to E(t)$ is strictly subhomogeneous in the sense that for any $\alpha \in (0, 1)$ and $\phi \in E(0)$ with $\phi \gg 0$, there holds $Q(t)(\alpha\phi) > \alpha Q(t)(\phi)$.

Indeed, for any $\phi \in E(0)$ with $\phi \not\equiv 0$ and $\alpha \in (0, 1)$, let $z(t, x) = v(t, x, \alpha\phi) - \alpha v(t, x, \phi)$. Then, $z(0, x) = 0$ and $z(t, x) \geq 0$ for all $t \geq 0$ and $x \in \bar{\Omega}$. We further show that $z(t, x) > 0$ for all $t > 0$ and $x \in \bar{\Omega}$. For simplicity, we let $f(t, x, v_1, v_2) := \sigma_2(t, x)(V^*(t, x) - v_2(t, x))v_1(t, x)$. It follows that

$$
\begin{aligned}
\frac{\partial z_2(t,x)}{\partial t} &= \frac{\partial v_2(t,x,\alpha\phi)}{\partial t} - \alpha\frac{\partial v_2(t,x,\phi)}{\partial t} \\
&= \nabla \cdot (\delta_2(t,x)\nabla v_2(t,x,\alpha\phi)) + f(t,x,v_1(t,x,\alpha\phi), v_2(t,x,\alpha\phi)) \\
&\quad - (\mu_1(t,x) + \mu_2(t,x)V^*(t,x))v_2(t,x,\alpha\phi) \\
&\quad - \alpha[\nabla \cdot (\delta_2(t,x)\nabla v_2(t,x,\phi)) \\
&\quad + f(t,x,v_1(t,x,\phi), v_2(t,x,\phi)) - (\mu_1(t,x) \\
&\quad + \mu_2(t,x)V^*(t,x))v_2(t,x,\phi)] \\
&= \nabla \cdot (\delta_2(t,x)\nabla z_2(t,x)) - (\mu_1(t,x) \\
&\quad + \mu_2(t,x)V^*(t,x))z_2(t,x) + h(t,x) \\
&\quad + \sigma_2(t,x)V^*(t,x))z_1(t,x) + \sigma_2(t,x)[-\alpha v_2(t,x,\phi)z_1(t,x)
\end{aligned}
$$

$$
\begin{aligned}
&\quad - v_1(t,x,\alpha\phi)z_2(t,x)] \\
&= \nabla\cdot(\delta_2(t,x)\nabla z_2(t,x)) - (\mu_1(t,x) \\
&\quad + \mu_2(t,x)V^*(t,x))z_2(t,x) + h(t,x) \\
&\quad + \sigma_2(t,x)(V^*(t,x) - \alpha v_2(t,x,\phi))z_1(t,x) \\
&\quad - \sigma_2(t,x)v_1(t,x,\alpha\phi)z_2(t,x) \\
&\geq \nabla\cdot(\delta_2(t,x)\nabla z_2(t,x)) - (\mu_1(t,x) + \mu_2(t,x)V^*(t,x))z_2(t,x) \\
&\quad - \sigma_2(t,x)v_1(t,x,\alpha\phi))z_2(t,x) + h(t,x),
\end{aligned}
$$

where $h(t,x) := f(t,x,\alpha v_1(t,x,\phi),\alpha v_2(t,x,\phi)) - \alpha f(t,x,v_1(t,x,\phi),v_2(t,x,\phi))$. Since the solution $v_1(t,x,\alpha\phi)$ is bounded, there exists a positive constant K such that

$$
\frac{\partial z_2(t,x)}{\partial t} \geq \nabla\cdot(\delta_2(t,x)\nabla z_2(t,x)) - Kz_2(t,x) + h(t,x). \tag{8}
$$

Let $\hat{T}(t,s) : \mathbb{Y} \to \mathbb{Y}$, $0 \leq s \leq t$, be the evolution operator of

$$
\begin{aligned}
&\frac{\partial u(t,x)}{\partial t} = \nabla\cdot(\delta_2(t,x)\nabla u(t,x)) - Ku(t,x), t > 0, x \in \Omega, \\
&\frac{\partial u(t,x)}{\partial \nu} = 0, t > 0, x \in \partial\Omega.
\end{aligned}
$$

Thus, the system

$$
\begin{aligned}
&\frac{\partial u(t,x)}{\partial t} = \nabla\cdot(\delta_2(t,x)\nabla u(t,x)) - Ku(t,x) + h(t,x), t > 0, x \in \Omega, \\
&\frac{\partial u(t,x)}{\partial \nu} = 0, t > 0, x \in \partial\Omega, \\
&u(0,x) = \varphi \in \mathbb{Y}, x \in \bar{\Omega},
\end{aligned} \tag{9}
$$

can be written as

$$
u(t,x,\varphi) = \hat{T}(t,0)(\varphi)(x) + \int_0^t \hat{T}(t,s)h(s,x)\mathrm{d}s, t \geq 0, x \in \bar{\Omega}, \varphi \in \mathbb{Y}. \tag{10}
$$

By Lemma 4, $v_1(t,x,\phi) > 0, \forall t > 0, x \in \bar{\Omega}$. It then follows that $h(t,x) = \sigma_2(t,x)\alpha \times v_1(t,x,\phi)(v_2(t,x,\phi) - \alpha v_2(t,x,\phi)) > 0$ for all $t > 0$ and $x \in \bar{\Omega}$. By equation (10) and the properties of $\hat{T}(t,s)$, we have for any $\varphi \geq 0$ with $\varphi \not\equiv 0$, the solution of system (9) satisfies $u(t,x,\varphi) > 0$ for all $t > 0$ and $x \in \bar{\Omega}$. Then by (8) and the comparison principle, we have $z_2(t,x) > 0$ for all $t > 0$ and $x \in \bar{\Omega}$. This implies that $v(t,\cdot,\alpha\phi) > \alpha v(t,\cdot,\phi), \forall t > 0$, that is, $Q(t)(\alpha\phi) > \alpha Q(t)(\phi)$ for all $t > 0$. Thus, for each $t > 0$, the map $Q(t)$ is strictly subhomogeneous.

By the above analysis and Lemma 4, it follows that $Q := Q(T)$ is a strongly monotone and strictly subhomogeneous map on $E(0)$. Since $Q(t)$ is compact for all $t > 0$, Q is asymptotically smooth on $E(0)$. Similarly, we see that P is also compact and strongly positive. By Zhao (2017b, Theorem 2.3.4 and Lemma 2.2.1), as applied to Q, we have the following threshold type result:

(i) If $r(P) \le 1$, then $(0, 0)$ is globally asymptotically stable for system (6) in $E(0)$.
(ii) If $r(P) > 1$, then system (6) has a unique positive T-periodic solution $v^*(t,x) = (H_i^*(t,x), V_i^*(t,x))$, and $v^*(t,x)$ is globally asymptotically stable for system (6) in $E(0) \setminus \{(0, 0)\}$.

In view of Lemma 2, we then have the desired threshold type result in terms of R_0. □

Next, we use the theory of chain transitive sets (see Zhao 2017b, Chapter 1) to lift the global stability result on system (6) to system (3).

Theorem 2 *Let (A1)–(A2) hold. The following statements are valid:*

(i) If $R_0 \le 1$, then the periodic solution $(0, 0, V^*(t,x))$ is globally asymptotically stable for system (3) in $X_1 := \{(\varphi_1, \varphi_2, \varphi_3) \in X : \varphi_3 \ne 0\}$.
(ii) If $R_0 > 1$, then system (3) admits a unique positive T-periodic solution $u^*(t,x) = (H_i^*(t,x), V_i^*(t,x), V^*(t,x))$, and $u^*(t,x)$ is globally asymptotically stable for system (3) in $X_2 := \{(\varphi_1, \varphi_2, \varphi_3) \in X : (\varphi_1, \varphi_2) \ne (0, 0)$ and $\varphi_3 \ne 0\}$.

Proof For any given $t \ge 0$, let $\tilde{Q}(t) : X \to X$ be the solution map of system (3), that is, $\tilde{Q}(t)(\varphi) = u(t, \cdot, \varphi)$, where $u(t, \cdot, \varphi)$ is the unique solution of system (3) with $u(0, \cdot, \varphi) = \varphi$. Then $\tilde{Q} := \tilde{Q}(T)$ is the Poincaré map associated with system (3). For any given $\bar{\varphi} \in X$ with $\bar{\varphi}_3(\cdot) \not\equiv 0$, let $\bar{u}(t,x) = (H_i(t,x), V_i(t,x), V(t,x))$ be the unique solution of system (3) with $\bar{u}(0, \cdot, \bar{\varphi}) = \bar{\varphi}$ and let $\omega(\bar{\varphi})$ be the omega limit set of the orbit $\{\tilde{Q}^n(\bar{\varphi})\}_{n \ge 0}$ for the discrete-time semiflow $\tilde{Q}^n$.

Since system (2) has a unique positive T-periodic solution $V^*(t,x)$ and it is globally stable, we have $\lim_{t\to\infty}(V(t,x) - V^*(t,x)) = 0$ uniformly for all $x \in \bar{\Omega}$ in $\mathbb{Y} \setminus \{0\}$. Then, $\lim_{n\to\infty}(\tilde{Q}^n(\bar{\varphi}))_3 = V^*(0, \cdot)$, where $(\tilde{Q}^n(\bar{\varphi}))_3$ is the third component of $\tilde{Q}^n(\bar{\varphi})$. Since $\tilde{Q}(t)$ is compact for each $t > 0$, $\omega(\bar{\varphi})$ is nonempty, compact and invariant for $\tilde{Q}$. Therefore, there exists a subset $\tilde{\omega}$ of $\mathbb{E}^+$ such that $\omega(\bar{\varphi}) = \tilde{\omega} \times \{V_0^*\}$, where $V_0^* = V^*(0, \cdot)$.

For any $\phi = (\phi_1, \phi_2, \phi_3) \in \omega(\bar{\varphi})$, there exists a sequence $n_k \to \infty$ such that $\tilde{Q}^{n_k}(\bar{\varphi}) \to \phi$ as $k \to \infty$. Since $V_i(n_k T, x) \le V(n_k T, x)$ for all $x \in \bar{\Omega}$, letting $n_k \to \infty$, we have $0 \le \phi_2(x) \le \phi_3(x) \equiv V_0^*$ for all $x \in \bar{\Omega}$. It then follows that $\tilde{\omega} \subset E(0)$. Clearly,

$$\tilde{Q}^n \mid_{\omega(\bar{\varphi})} (\phi_1, \phi_2, V_0^*) = Q^n \mid_{\tilde{\omega}} (\phi_1, \phi_2) \times \{V_0^*\}, \forall (\phi_1, \phi_2) \in \tilde{\omega}, n \ge 0.$$

Since $\omega(\bar{\varphi})$ is an internally chain transitive set for $\tilde{Q}$ on X, we can easily check that $\tilde{\omega}$ is an internally chain transitive set for Q on $E(0)$.

In the case where $R_0 \le 1$, Theorem 1 (i) implies that $(0, 0)$ is globally asymptotically stable for Q in $E(0)$. It follows from Zhao (2017b, Theorem 1.2.1) that $\tilde{\omega} = \{(0, 0)\}$, and hence, $\omega(\bar{\varphi}) = \{(0, 0, V_0^*)\}$. This implies that statement (i) is valid.

In the case where $R_0 > 1$, it follows from Theorem 1 (ii) and (Zhao 2017b, Theorem 1.2.2) that either $\tilde{\omega} = \{(0, 0)\}$ or $\tilde{\omega} = \{(H_{i0}^*, V_{i0}^*)\}$, where $H_{i0}^* = H_i^*(0, \cdot)$ and $V_{i0}^* = V_i^*(0, \cdot)$. Next we claim that $\tilde{\omega} \ne \{(0, 0)\}$. Assume, by contradiction, that

$\tilde{\omega} = \{(0,0)\}$, then we have $\omega(\bar{\varphi}) = \{(0, 0, V_0^*)\}$. Thus, $\lim_{t\to\infty}(H_i(t,x), V_i(t,x)) = (0,0)$ and $\lim_{t\to\infty} (V(t,x) - V^*(t,x)) = 0$ uniformly for $x \in \bar{\Omega}$. In view of system (3), it then follows that for any $\epsilon > 0$, there exists $t_0 = t_0(\epsilon) > 0$ such that $\|V(t,x) - V^*(t,x)\| < \epsilon$ and $\|(V(t,x) - V_i(t,x)) - V^*(t,x)\| < \epsilon$ for all $t \geq t_0$ and $x \in \bar{\Omega}$. Then, for any $t \geq t_0$, we see from system (3) that

$$\begin{aligned}
\frac{\partial H_i(t,x)}{\partial t} &= \nabla \cdot (\delta_1(t,x)\nabla H_i(t,x)) - \lambda(t,x)H_i(t,x) + \sigma_1(t,x)H_u(x)V_i(t,x),\\
\frac{\partial V_i(t,x)}{\partial t} &\geq \nabla \cdot (\delta_2(t,x)\nabla V_i(t,x)) + \sigma_2(t,x)(V^*(t,x)-\epsilon)H_i(t,x)\\
&\quad - \mu_1(t,x)V_i(t,x) - \mu_2(t,x)(V^*(t,x)+\epsilon)V_i(t,x). \qquad (11)
\end{aligned}$$

Let r_ϵ be the spectral radius of the Poincaré map associated with the following periodic linear system:

$$\begin{aligned}
\frac{\partial H_i(t,x)}{\partial t} &= \nabla \cdot (\delta_1(t,x)\nabla H_i(t,x)) - \lambda(t,x)H_i(t,x)\\
&\quad + \sigma_1(t,x)H_u(x)V_i(t,x),\ t>0,\ x\in\Omega,\\
\frac{\partial V_i(t,x)}{\partial t} &= \nabla \cdot (\delta_2(t,x)\nabla V_i(t,x)) + \sigma_2(t,x)(V^*(t,x)-\epsilon)H_i(t,x)\\
&\quad - \mu_1(t,x)V_i(t,x)\\
&\quad - \mu_2(t,x)(V^*(t,x)+\epsilon)V_i(t,x),\ t>0,\ x\in\Omega,\\
\frac{\partial H_i}{\partial \nu} &= \frac{\partial V_i}{\partial \nu} = 0,\ t>0,\ x\in\partial\Omega. \qquad (12)
\end{aligned}$$

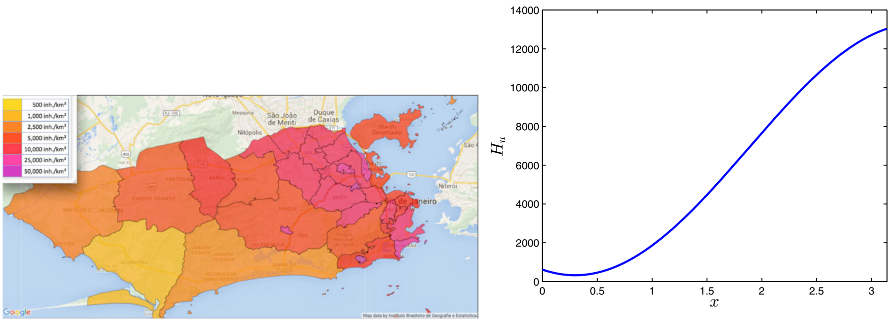
Since $\lim_{\epsilon\to\infty} r_\epsilon = r(P) > 1$, we can fix $0 < \epsilon < \min_{t\in[0,T],x\in\bar{\Omega}} V^*(t,x)$ small enough such that $r_\epsilon > 1$. By the arguments similar to those for Liang et al. (2017, Theorem 2.16), there exists a positive T-periodic function $v_\epsilon^*(t,x)$ such that $e^{\mu_\epsilon t}v_\epsilon^*(t,x)$ is a solution of system (12), where $\mu_\epsilon = \frac{\ln r_\epsilon}{T} > 0$. Since $\bar{\varphi} \in X_2$, it follows from the proof of Lemma 4 that $H_i(t,\cdot) \gg 0$ and $V_i(t,\cdot) \gg 0$ for all $t > 0$. Then, there exist a large integer $n_0 > 0$ and a small number $\alpha > 0$ such that $n_0T \geq t_0$ and $(H_i(n_0T,\cdot), V_i(n_0T,\cdot) \geq \alpha e^{\mu_\epsilon n_0 T} v_\epsilon^*(0,\cdot)$. By the comparison theorem, it follows that

$$(H_i(t,\cdot), V_i(t,\cdot)) \geq \alpha e^{\mu_\epsilon t} v_\epsilon^*(t,\cdot) \gg 0,\ \ \forall t \geq n_0T,$$

which contradicts $\lim_{t\to\infty}(H_i(t,x), V_i(t,x)) = (0,0)$ uniformly for all $x \in \bar{\Omega}$. Thus, $\tilde{\omega} = (H_{i0}^*, V_{i0}^*)$, and hence, $\omega(\bar{\varphi}) = \{(H_{i0}^*, V_{i0}^*, V_0^*)\}$. This implies that statement (ii) is valid. □

4 Numerical Simulations

In this section, we present some numerical simulations to investigate the impact of seasonality and spatial heterogeneous infection on the Zika transmission. We apply



(a) **(b)**

Fig. 1 (Color figure online) **a** Population density of Rio de Janeiro Municipality sub-districts (Fitzgibbon et al. 2017). . Source: https://www.citypopulation.de/php/brazil-rio.php. **b** Density of susceptible host $H_u(x)$ in Rio de Janeiro Municipality

system (3) to Rio de Janeiro Municipality, Brazil, which has a resident population of 6, 718, 903 and a density of 5, 598 inhabitants per square kilometer (see, https://www.citypopulation.de/php/brazil-regiaosudeste-admin.php?adm2id=3304557). The first confirmed case is reported in May 2015 and the main vector is the *Aedes aegypti* mosquito. From the map of Rio de Janeiro Municipality (see Fig. 1a), we see that the east–west is much longer than north–south and the eastern region has a highest population density. The sub-district population densities range from about 1, 000 to 50, 000 per square kilometer. For simplicity, we then focus on one dimensional domain Ω, which can be chosen $\Omega = (0, \pi)$, without loss of generality. The density of susceptible host is assumed to be the function $H_u(x) = 7405 \times (1.05 - \cos(x - 0.48))$ (see Fig. 1b), which corresponds approximately to the total population density in Fig. 1a. Assuming that there are 10 mosquitoes per host, we have the maximum carrying capacity for vector $N_m(x) = 10 \times H_u(x)$. Since Brazilian average life expectancy is 76 years, we can estimate the natural death rate of host as $1/(76 \times 12)$ Month^{-1}.

Since host components of the transmission cycle do not undergo a significant seasonal variation, we use constant values to describe them. The majority of the parameters associated with the *Aedes aegypti* mosquitoes are influenced by temperature such as the biting rate, the vector breeding rate, the mortality rate of vector, the carrying capacity, and so on. We list the temperature-dependent functions and the values for temperature-independent rates in Table 1 by using some published data. Additionally, according to Huber et al. (2018), Palamara et al. (2014), the temperature-dependent carrying capacity is modeled as

$$K(C) = \left(1 - \frac{\mu_1(C_0)}{\beta(C_0)}\right) \times N_m \times e^{\frac{-0.5(C-C_0)^2}{8.617\times 10^{-5}(C+273)(C_0+273)}},$$

where the reference temperature $C_0 = 29°$C. Then, we evaluate the time-dependent parameters by using the monthly mean temperature for Rio de Janeiro according to the Climate Change Knowledge Portal (https://climateknowledgeportal.worldbank.org/). Set the period $T = 12$ Months. Thus, using the curve fitting tool (CFTOOL) in

Table 1 Values (ranges) for constant parameters and functions of temperature C in °C for system (3)

Parameter	Description	Constant/formula	References
λ	Loss rate of infected host	$1/(76 \times 12) + 0.07 \times 30.4$ Month^{-1}	Shutt et al. (2017)
$b(C)$	Biting rate	$2.02 \times 10^{-4} C(C - 13.35)(40.08 - C)^{1/2} \times 30.4$ Month^{-1}	Mordecai et al. (2017)
$\beta_{vh}(C)$	Transmission probability per bite from infectious vector to host	$8.49 \times 10^{-4} C(C - 17.05)(35.83 - C)^{1/2}$	Mordecai et al. (2017)
$\beta_{hv}(C)$	Transmission probability per bite from infectious host to vector	$4.91 \times 10^{-4} C(C - 12.22)(37.46 - C)^{1/2}$	Mordecai et al. (2017)
$\mu_1(C)$	Natural mortality rate of vector	$\frac{1}{-0.148(C-9.16)(C-37.73)} \times 30.4$ Month^{-1}	Mordecai et al. (2017)
$\beta(C)$	Breeding rate of vector	$\frac{-4.0302\times 10^{-9}}{\mu_1(C)} C^2(C - 14.58)(C - 13.56)(C - 11.36) \times (C - 38.29)(39.17 - C)^{1/2} \times 30.4$ Month^{-1}	Mordecai et al. (2017)
$\mu_2(C)$	Density-dependent vector loss rate	$\frac{\beta(C)-\mu_1(C)}{K(C)}$ Mosquitoes$^{-1}\times$ Month^{-1}	Charron et al. (2013)
δ_1	Host diffusion coefficient	1×30.4 km^2/Month	Lou and Zhao (2011)
δ_2	Vector diffusion coefficient	$1.25 \times 10^{-2} \times 30.4$ km^2/Month	Lou and Zhao (2011)

Table 2 Monthly mean temperatures for Rio de Janeiro (in °C)

Month	January	February	March	April	May	June
Temperature	25.65	26.03	25.31	23.61	21.67	20.52
Month	July	August	September	October	November	December
Temperature	20.20	20.72	21.29	22.36	23.40	24.71

MATLAB and Tables 1 and 2, we can fit the time periodic functions for parameters to obtain

$$
\begin{aligned}
b(t) =& (0.1841 + 0.05958\cos(\pi t/6) + 0.01361\sin(\pi t/6)\\
&- 3.333\times 10^{-5}\cos(2\pi t/6) + 0.009324\sin(2\pi t/6)\\
&- 0.0012\cos(3\pi t/6) - 0.0006667\sin(3\pi t/6)\\
&+ 5\times 10^{-5}\cos(4\pi t/6) - 0.0001155\sin(4\pi t/6) + 0.0003714\cos(5\pi t/6)\\
&- 0.0006242\sin(5\pi t/6) - 0.0007667\cos(6\pi t/6)) \times 30.4\ \text{Month}^{-1},\\
\beta_{vh}(t) =& 0.4098 + 0.1947\cos(\pi t/6) + 0.04374\sin(\pi t/6) - 0.002083\cos(2\pi t/6)\\
&+ 0.02933\sin(2\pi t/6) - 0.0041\cos(3\pi t/6) - 0.003133\sin(3\pi t/6)\\
&+ 0.0003667\cos(4\pi t/6) - 0.0004907\sin(4\pi t/6) + 0.001362\cos(5\pi t/6)\\
&- 0.002019\sin(5\pi t/6) - 0.002567\cos(6\pi t/6),\\
\beta_{hv}(t) =& 0.4583 + 0.1279\cos(\pi t/6) + 0.02857\sin(\pi t/6) - 0.001792\cos(2\pi t/6)\\
&+ 0.0191\sin(2\pi t/6) - 0.002567\cos(3\pi t/6) - 0.002133\sin(3\pi t/6)\\
&+ 0.0002417\cos(4\pi t/6) - 0.0002742\sin(4\pi t/6)\\
&+ 0.0009287\cos(5\pi t/6)\\
&- 0.001306\sin(5\pi t/6) - 0.001683\cos(6\pi t/6),\\
\mu_1(t) =& (0.03384 - 0.0004271\cos(\pi t/6) + 9.33\times 10^{-5}\sin(\pi t/6)\\
&+ 0.0005917\cos(2\pi t/6)\\
&+ 0.0002165\sin(2\pi t/6) - 5\times 10^{-5}\cos(3\pi t/6) + 0.0002\sin(3\pi t/6)\\
&- 4.166\times 10^{-5}\cos(4\pi t/6) - 4.329\times 10^{-5}\sin(4\pi t/6)\\
&- 2.292\times 10^{-5}\cos(5\pi t/6)\\
&+ 6.7\times 10^{-6}\sin(5\pi t/6) + 8.341\times 10^{-6}\cos(6\pi t/6)) \times 30.4\ \text{Month}^{-1},\\
\beta(t) =& (16.77 + 8.093\cos(\pi t/6) + 1.92\sin(\pi t/6) + 0.2185\cos(2\pi t/6)\\
&+ 1.348\sin(2\pi t/6) - 0.2163\cos(3\pi t/6) - 0.05037\sin(3\pi t/6)\\
&+ 0.004967\cos(4\pi t/6) - 0.05488\sin(4\pi t/6)\\
&+ 0.0463\cos(5\pi t/6) - 0.08378\sin(5\pi t/6)\\
&- 0.09837\cos(6\pi t/6))30.4\ \text{Month}^{-1},\\
K(t) =& (0.1804 + 0.2413\cos(\pi t/6) + 0.07915\sin(\pi t/6) + 0.07087\cos(2\pi t/6)\\
&+ 0.07705\sin(2\pi t/6) - 0.00275\cos(3\pi t/6) + 0.02712\sin(3\pi t/6)
\end{aligned}
$$

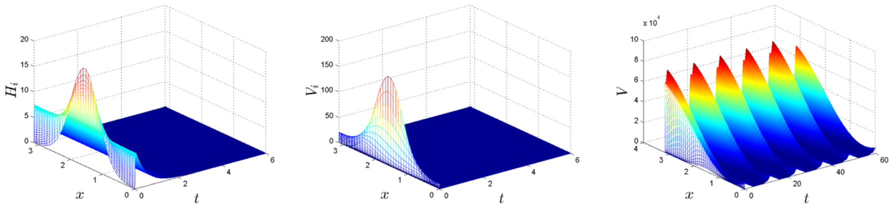


Fig. 2 (Color figure online) Evolution of $H_i(t, x)$, $V_i(t, x)$ and $V(t, x)$ when $R_0 < 1$

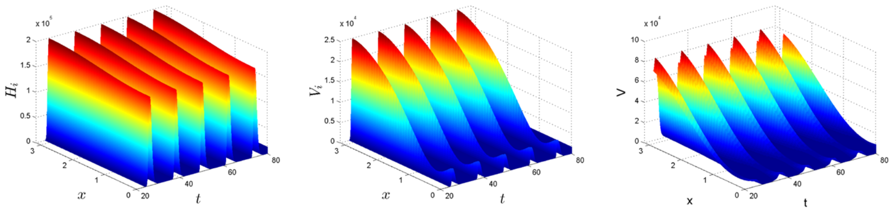


Fig. 3 (Color figure online) Evolution of $H_i(t, x)$, $V_i(t, x)$ and $V(t, x)$ when $R_0 > 1$

$$
\begin{aligned}
&- 0.007217 \cos(4\pi t/6) - 0.001559 \sin(4\pi t/6) - 0.001664 \cos(5\pi t/6) \\
&- 0.004829 \sin(5\pi t/6) - 0.0009667 \cos(6\pi t/6)) \times N_m,
\end{aligned}
$$

$$
\mu_2(t) = \frac{\beta(t) - \mu_1(t)}{K(t)} \text{ Month}^{-1}.
$$

For time-periodic systems, it is hard to obtain an explicit formulation of the basic reproduction ratio, but we can numerically compute it. We use the numerical algorithm for the computation of R_0 developed in Liang et al. (2019, Lemma 2.5 and Remark 3.2). With this set of parameters, we numerically calculate the basic reproduction ratio to obtain $R_0 = 0.1353 < 1$. We use the backward difference method on time variable t and the central difference method on space variable x to simulate the solutions of system (3). Choosing the initial data as

$$
\begin{aligned}
H_i(0, x) &= \frac{1}{0.38\sqrt{2\pi}} e^{-\frac{(x-\pi/2)^2}{2\times 0.38^2}} \times 20, \quad V_i(0, x) = \frac{1}{0.38\sqrt{2\pi}} e^{-\frac{(x-\pi/2)^2}{2\times 0.38^2}} \times 200, \\
V(0, x) &= \frac{1}{0.38\sqrt{2\pi}} e^{-\frac{(x-\pi/2)^2}{2\times 0.38^2}} \times 2000, \quad \forall x \in [0, \pi].
\end{aligned}
$$

Figure 2 shows the long-term behavior of system (3). The densities of infectious host and vector both go to zero, and the density of total vector stabilizes at a positive periodic solution, which implies that the disease will be eliminated. If the biting rate $a(t)$ increases to $8.5a(t)$, we obtain $R_0 = 1.1503 > 1$. In this case, the solution converges to a positive periodic solution eventually (see Fig. 3), which means that the disease will persist and exhibit periodic fluctuation eventually. This is consistent with Theorem 2.

For the purpose of exploring control measures, it is necessary to know the relative importance of each factor responsible for the transmission of disease. Since R_0 mea-

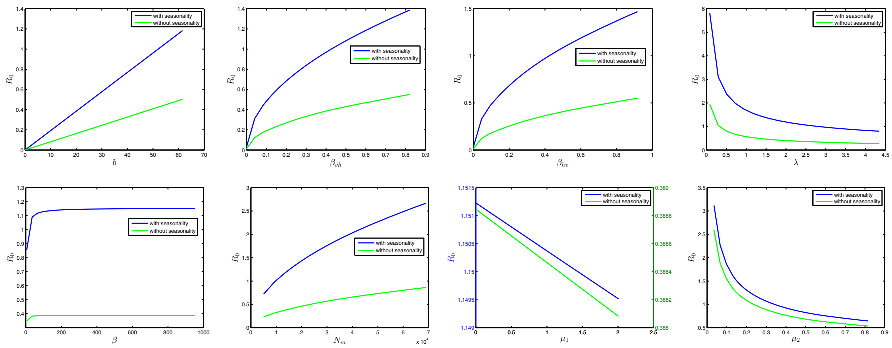


Fig. 4 (Color figure online) Relationships of R_0 and b, β_{vh}, β_{hv}, λ, β, N_m, μ_1 and μ_2 with and without seasonality, respectively

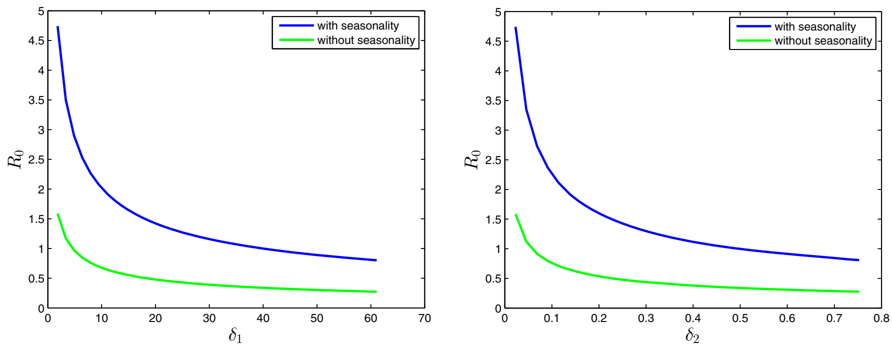


Fig. 5 (Color figure online) Effects of population diffusion

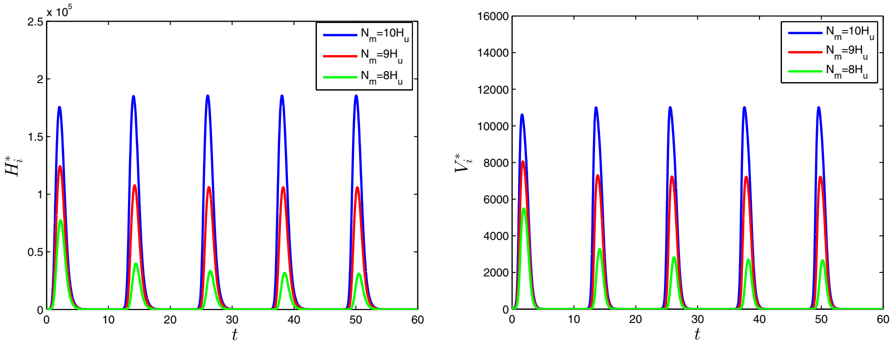


Fig. 6 (Color figure online) x-intersections of numerical periodic solutions $H_i^*(t, x)$ and $V_i^*(t, x)$ at location $x = \pi/2$

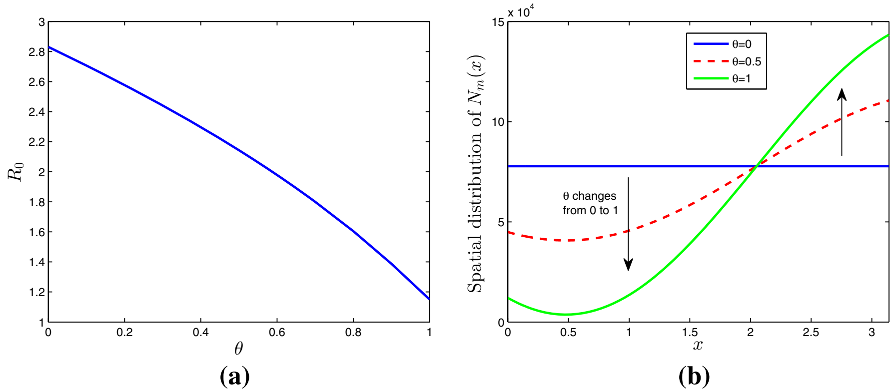


Fig. 7 (Color figure online) **a** R_0 as a function of θ. **b** The spatial distribution of $N_m(x)$

sures the risk of an epidemic, we numerically explore the relationship between R_0 and some coefficients in our model system. In the following, we keep the parameter values the same as those in Fig. 3. Figure 4 shows that R_0 is an increasing function with respect to parameters b, β_{vh}, β_{hv}, β, and N_m, but R_0 is a decreasing function of λ, μ_1 and μ_2, respectively. From the seventh of Fig. 4, there is a very small change in R_0 when μ_1 increases under seasonal heterogeneity or homogeneity. That is, μ_1 is lowly correlated with R_0. Other parameters with a broad range of variability would be expected to be stronger drivers of change in R_0. In order to control the spread of disease, the ideal situation is to reduce R_0 to be less than unity. Moreover, Fig. 5 shows that R_0 decreases as δ_i, $i = 1, 2$, increases, but we find that there is a sharp decline in R_0 when δ_i is small and then R_0 is decreasing very slowly. This means that it does not seem like a good control strategy by increasing the host or vector population mobility. Moreover, to study the impact of the seasonal heterogeneity on the Zika transmission, the time-averaged parameter is assumed by $[g] := \frac{1}{T}\int_0^T g(t)\mathrm{d}t$. In Figs. 4 and 5, the green curves refer system (3) is under time-averaged parameters and the blue ones refer system (3) with time-periodic parameters. It is easy to see that the green curves all lie below the blue ones, respectively. These indicate that the disease risk will be underestimated if we ignore the seasonality of Zika virus (see Fig. 6).

We also compare the infectious host and vector population sizes under different maximum carrying capacities in vectors. This implies that for the lower maximum of carrying capacity, there is a smaller epidemic peak. To explore the spatial heterogeneity effect on R_0, we can assume that the maximum vector carrying capacity $N_m = 74050\times(1.05 - \theta\cos(x - 0.48))$ with $\theta \in [0, 1]$. R_0 is an increasing function with respect to θ (see Fig. 7a). Note that the vector is distributed homogeneously in Rio de Janeiro Municipality when $\theta = 0$. But the host distribution is concentrated in the east of Rio de Janeiro Municipality. In this case, the value of R_0 is big and there is a higher disease risk. More and more vectors are concentrated in humans crowded places as θ increases from 0 to 1. In this case, the value of R_0 will decrease as θ increases. These show that the spatial heterogeneities in host and vector distributions impact the spread of Zika virus.

5 Discussion

In this paper, we have investigated a periodic Zika transmission model which takes into account the seasonality and spatial heterogeneity in host and vector populations. Applying the theory developed in Liang et al. (2019), we have derived the basic reproduction ratio R_0. Using the theories of monotone dynamical systems and chain transitive sets, we have obtained a threshold type result on the global stability for the model system in terms of R_0, that is, if $R_0 \leq 1$, then the disease will die out; if $R_0 > 1$, then the disease will eventually stabilize at a positive periodic state.

In order to discuss the effect of the temperature on Zika transmission, we have used some published data and temperature-dependent parameters with functional forms to simulate the Zika transmission case in Rio de Janeiro Municipality, Brazil. We simulated the long-time behaviors of system to verify our analytic results. The value of R_0 measures the risk of an epidemic or pandemic in infectious diseases. Then, we numerically explored the influence of some parameters on R_0. As shown in Figs. 4 and 5, reducing the biting rate, the transmission probability from infectious host (vector) to vector (host), the vector breeding rate, the maximum carrying capacity in vector, and increasing the host recovery rate, the vector mortality rate, the host and vector diffusion coefficients can reduce the risk of disease. This suggests that vector control strategies play a very important role in the Zika transmission, especially, reducing the vector population size and biting rate. Thus, we may provide some corresponding potential control strategies: the biting rate can be reduced by avoiding the exposure to mosquitoes through personal protection including using window screens, door screens and mosquito nets, and using repellents on exposed skin or clothing; the transmission probability per bite from infectious vector to host can be decreased by using transmission-blocking strategies; the breeding rate of vector can be reduced by using larvicides; the maximum carrying capacity in vector can be decreased by eliminating vector breeding sites near human habitats such as water storage, ponds and puddles; the mortality rate of vector can be increased by using adulticides; the recovery rate of host can be increased by the prompt diagnosis and medication treatments. In addition, the vaccines against Zika virus seem the best way to protect humans over the long term, but it remains an urgent need to develop a Zika vaccine (Makhluf and Shresta 2018). Figures 4 and 5 show the ignorance of seasonality in the Zika virus transmission may underestimate the value of R_0.

A larger maximum of carrying capacity in vectors leads to an earlier peak and larger epidemic peak. This allows us to highlight the importance of the maximum of carrying capacity in vectors and its influence on the spread of Zika virus. We also found that the spatial heterogeneities of the maximum of carrying capacity in vectors decease the disease risk.

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