Assignment 4, MATH 6104

1. Let $\{Q_t\}_{t\geq 0}$ be the solution semiflow on $\mathcal{C}_1$ of the Fisher equation $u_t = u_{xx} + u(1-u)$. Show that for each $t > 0$, the map $Q := Q_t$ satisfies assumptions (A1), (A2), (A3), (A4) and (A5).

2. Let $c > 0$ be a real number and $g(t)$ be a continuous and bounded function defined on $\mathbb{R}$. Prove that the second order ordinary differential equation $cu'(t) = u''(t) - u(t) + g(t)$ has a unique bounded solution $u^*(t)$ defined on $\mathbb{R}$, and also give an explicit expression of $u^*(t)$.

3. Use the limiting argument to prove that the Fisher equation $u_t = u_{xx} + u(1-u)$ has a monotone and positive traveling wave solution $U(x+2t)$ with $U(-\infty) = 0$ and $U(+\infty) = 1$.

4. Use the linearization method to prove that for any $c \in (0, 2)$, the Fisher equation $u_t = u_{xx} + u(1-u)$ has no positive traveling wave solution $U(x+ct)$ with $U(-\infty) = 0$.

5. Find sufficient conditions on $f \in C^1(\mathbb{R}, \mathbb{R})$ such that the spatially homogeneous equation $u' = f(u)$ admits the bistable dynamics, and then discuss the sign of the wave speed c of the bistable traveling wave solution $U(x-ct)$ to the reaction-diffusion equation $u_t = u_{xx} + f(u)$.