Assignment 1, MATH 6104

1. Let (X, d) be a metric space and $f : X \to X$ be a continous map. Prove that the omega limit set of any precompact forward orbit is compact and invariant for f.

2. Let $\Phi(t) : X \to X$ be a continuous autonomous semiflow. Assume that $\Phi(t)$ admits a global attractor A on X, and for any $T > 0$, the map $\Phi(T) : X \to X$ has a fixed point. Prove that $\Phi(t) : X \to X$ has an equilibrium $x_0 = \Phi(t)x_0, \ \forall t \geq 0$.

3. Let M_0 be an open subset of a complete metric space (M, d) and let $\partial M_0 := M \setminus M_0$. Define $\rho(x) = d\left(x, \partial M_0\right), \forall x \in M$, and

$$d_0(x, y) = \left|\frac{1}{\rho\left(x\right)} - \frac{1}{\rho\left(y\right)}\right| + d(x, y), \ \forall x, y \in M_0.$$

Prove that $d_0(x, y)$ is a metric function on M_0 and (M_0, d_0) is a complete metric space.

4. Prove that a global attractor for a semiflow on a finite dimensional space attracts any bounded set.

5. Use the persistence theory to find a set of sufficient conditions for two species competitive ODE system

$$\begin{aligned} \frac{du_1}{dt} &= u_1(b_1 - a_{11}u_1 - a_{12}u_2) \\ \frac{du_2}{dt} &= u_2(b_2 - a_{21}u_1 - a_{22}u_2) \end{aligned}$$

to be uniformly persistent on $\text{Int}(\mathbb{R}^2_+)$ and to have at least one positive equilibrium.