PERGAMON

Nonlinear Analysis 47 (2001) 1105–1116

Nonlinear Analysis

www.elsevier.nl/locate/na

The simplest normal forms associated with a triple zero eigenvalue of indices one and two[1]

P. Yu[2] and Y. Yuan

Department of Applied Mathematics, University of Western Ontario, London, Ontario, Canada N6A 5B7

Abstract

This paper considers the computation of the simplest normal forms of differential equations, with the particular attention on the systems whose Jacobian evaluated at an equilibrium has a triple zero eigenvalue of index one or two. The computation is based on near identity nonlinear transformations. Explicit formulas are derived for computing the simplest normal forms and the associated nonlinear transformations, which facilitates a direct implementation on computer systems using Maple.

Key words: Differential equation, Conventional normal form, The simplest normal form, Nonlinear transformation, Computer algebra
PACS: 34C20, 58F36, 34A34

1 Introduction

Normal form theory plays an important role in the study of differential equations related to complex behavior patterns such as bifurcation and instability [1–3]. It provides a convenient tool for computing a simple form of the original differential equations. The basic idea of the normal form theory is employing successive, near identity nonlinear transformations to obtain a simpler form. Many research results related to this area may be found from the above mentioned references.

It is well known that in general normal forms are not uniquely defined, and in fact further reductions on several cases of convention normal forms (CNF) have been discussed. Ushiki [4] introduced the method of infinitesimal deformation associated with Lie bracket to obtain simpler forms than CNF. Baider and Churchill [5] developed grading functions based on Lie algebra to define the first, the second, etc., order normal forms. They considered Hopf type

[1] Supported by the Natural Science and Engineering Research Council of Canada.
[2] Corresponding author. Email address: pyu@pyu1.apmaths.uwo.ca

PII: S0362-546X(01)00250-4

singularity and obtained the simplest "form". This case was recently reinvestigated from the computation point of view, and explicit formulations were obtained for computing the coefficients of the simplest normal form (SNF) and the nonlinear transformation [6]. The further reduction of the normal forms for the Bodganov-Takens singularity has been studied by Baider and Sanders [7], Kokubu et al. [8] and Wang et al. [9] using the Lie bracket method. Sanders and Meer [10] also used the same method to study the Hamiltonian 1:2 resonant case and showed that the second order normal form is "unique". Here, the "unique" means that the normal form is "minimum" or "simplest". More references can be found from the tutorial articles [11,12]. In the CNF theory, by saying that "normal forms are not unique" it usually implies at least one of the two cases: (1) For a same system, its normal forms may have different "form"; (2) Even for a same "form", the coefficient of the CNF may be different. For example, a system associated with Bodganov-Takens singularity may typically have two forms [1]; while the normal forms of Hopf bifurcation have the same form with only odd order terms presented, but their coefficients may be different [6]. Thus, certain methods such as adjoint operator approach [13] were developed to determine a *unique* normal form. But such a unique normal form is not the *simplest* form, and thus we prefer to use the simplest normal form (SNF), which means that the total number of the SNF up to certain order cannot be further simplified by any other nonlinear transformations.

Recently, many researchers have paid attention to the computation of the SNF and efficient computation methods using computer algebra systems have been developed (e.g., see [6,14–18]). In this paper, a method is presented to compute the SNF of differential equations associated with a triple zero eigenvalue of index one and two. Our method is based on nonlinear transformations. The key step in the computation is to find an appropriate pattern of nonlinear transformations so that the resulting normal form is the simplest. The main results are obtained using matrix theory and algebraic computations, which can provide a direct guideline for developing computer software. In fact, based on the explicit formulas derived in this paper, Maple has been used to develop symbolic programs.

The general formulas for commutating the SNF of differential equations are given in the next section. The detailed results are presented in Section 3, and conclusions are drawn in Section 4.

2 General formulation of the SNF

Consider the general system, described by

$$\dot{\boldsymbol{x}} = J\boldsymbol{x} + \boldsymbol{f}(\boldsymbol{x}), \quad \boldsymbol{x} \in \boldsymbol{R}^N, \tag{1}$$

which has an equilibrium at the origin $\mathbf{0}$, and J is the Jacobian matrix and assumed, without loss of generality, in a standard Jordan canonical form. Further, it is assumed that all eigenvalues of J have zero real parts. In other words, the dynamics of system (1) is described in an N-dimensional center manifold. In general, we may write system (1) in a vector form

$$\boldsymbol{v} = [J + \{f_1(\boldsymbol{x}),\, f_2(\boldsymbol{x}),\, \ldots,\, f_N(\boldsymbol{x})\}]\,\boldsymbol{\partial} \tag{2}$$

where the differentiation operator $\boldsymbol{\partial}$ is defined as $\boldsymbol{\partial} = (\partial_{x_1},\, \partial_{x_2},\, \ldots,\, \partial_{x_N})^T$, and $\boldsymbol{f} = (f_1,\, f_2,\, \ldots,\, f_N)^T$ is assumed to be a nonlinear analytic function. Thus, the functions $\boldsymbol{f}$ can be expanded into the vector homogeneous polynomials of $\boldsymbol{x}$ near the equilibrium $\boldsymbol{x} = \mathbf{0}$. Then we define the linear vector space $\mathcal{H}_n$ which consists of the nth degree homogeneous vector polynomials $\boldsymbol{f}^n(\boldsymbol{x})$. Further one may use the linear part $\boldsymbol{v_1} = J\,\boldsymbol{x}$ to define the homological operator L_n as

$$\begin{aligned} &L_n :\ \mathcal{H}_n \mapsto \mathcal{H}_n \\ &U_n \in \mathcal{H}_n \mapsto L_n(U_n) = [U_n,\, \boldsymbol{v_1}] \in \mathcal{H}_n \end{aligned} \tag{3}$$

where the operator $[U_n, \boldsymbol{v_1}]$ is called Lie bracket, defined by

$$[U_n,\, \boldsymbol{v_1}] = D\boldsymbol{v_1} \cdot U_n - DU_n \cdot \boldsymbol{v_1}. \tag{4}$$

Next, we define the space $\mathcal{R}_n$ as the range of L_n, and the complementary space of $\mathcal{R}_n$ as $\mathcal{K}_n = Ker\,(L_n)$. Thus,

$$\mathcal{H}_n = \mathcal{R}_n \oplus \mathcal{K}_n, \tag{5}$$

and we can then choose basis for the spaces $\mathcal{R}_n$ and $\mathcal{K}_n$. Consequently, a vector homogeneous polynomial $\boldsymbol{f}^n(\boldsymbol{x}) \in \mathcal{H}_n$ can be split into two parts: one can be spanned by the basis of $\mathcal{R}_n$ and the other by that of $\mathcal{K}_n$.

To find the SNF and the associated nonlinear transformation, one can start from the quadratic (2nd-order) terms: For $n = 2$, define

$$\begin{aligned} &Y_2 = \{\textstyle\sum d_{1\alpha_2}\, \boldsymbol{x}^{\alpha_2},\ \sum d_{2\alpha_2}\, \boldsymbol{x}^{\alpha_2},\ \ldots,\ \sum d_{N\alpha_2}\, \boldsymbol{x}^{\alpha_2}\}^T, \\ &V_2 = \boldsymbol{f}^2(\boldsymbol{x}) \equiv \{\textstyle\sum p_{1\alpha_2}\, \boldsymbol{x}^{\alpha_2},\ \sum p_{2\alpha_2}\, \boldsymbol{x}^{\alpha_2},\ \ldots,\ \sum p_{N\alpha_2}\, \boldsymbol{x}^{\alpha_2}\}^T, \end{aligned} \tag{6}$$

where the α denotes taking summation of all possible non-negative integers $k_1,\, k_2,\, \ldots,\, k_N$ such that $k_1 + k_2 + \ldots + k_N = n$, and we say that α has index n, denoted by α_n. Then $\boldsymbol{x}^{\alpha_n} = x_1^{k_1}\, x_2^{k_2} \ldots x_N^{k_N}$. Note that Y_2 denotes the general second-order nonlinear transformation while V_2 represents the terms of the vector field $\boldsymbol{f}$ up to second order. Further let

$$\begin{aligned} &W = V_2 + [\,Y_2, \boldsymbol{v_1}\,], \\ &U_2 = \boldsymbol{v_1} + \{\textstyle\sum q_{1\alpha_2}\, \boldsymbol{x}^{\alpha_2},\ \sum q_{2\alpha_2}\, \boldsymbol{x}^{\alpha_2},\ \ldots,\ \sum q_{N\alpha_2}\, \boldsymbol{x}^{\alpha_2}\}^T, \end{aligned} \tag{7}$$

where U_2 denotes the SNF up to second order. It should be noted that the form of U_2 is same as that of V_2 since the SNF is at least as simple as the original vector fields.

Now setting $W - U_2 = 0$, and then balancing the coefficients of the second-order terms yields a set of algebraic equations which can be used to determine the coefficients of the nonlinear transformation, d_{ijkl}, and the coefficients of the SNF, q_{jkl}. It should be pointed out that in general the number of the coefficients is larger than the number of the algebraic equations. Thus, some of the coefficients of the nonlinear transformation cannot be determined. In the CNF theory, the undetermined coefficients are set zero and therefore the nonlinear transformation is simplified. However, in order to further simplify the normal form, we should not set the undetermined coefficients zero but let them carry over to higher order equations and hope that they can be used to simplify higher order normal forms. This is the fundamental difference between the CNF and the SNF.

Similarly, we can carry out the above procedure to $n = 3, 4, \dots,$ and in general we may obtain

$$Y_n = \{ \sum d_{1\alpha_n} \boldsymbol{x}^{\alpha_n}, \ \sum d_{2\alpha_n} \boldsymbol{x}^{\alpha_n}, \ \dots, \ \sum d_{N\alpha_n} \boldsymbol{x}^{\alpha_n} \}^T,$$

$$V_n = \{ \sum p_{1\alpha_n} \boldsymbol{x}^{\alpha_n}, \ \sum p_{2\alpha_n} \boldsymbol{x}^{\alpha_n}, \ \dots, \ \sum p_{N\alpha_n} \boldsymbol{x}^{\alpha_n} \}^T,$$

$$U_n = \{ \sum q_{1\alpha_n} \boldsymbol{x}^{\alpha_n}, \ \sum q_{2\alpha_n} \boldsymbol{x}^{\alpha_n}, \ \dots, \ \sum q_{N\alpha_n} \boldsymbol{x}^{\alpha_n} \}^T,$$

$$W = [Y_n, \, \boldsymbol{v}_1] + \sum_{i=2}^{n-1} \left\{ [Y_{n-i+1}, \, V_i] + DY_i \, (\boldsymbol{f}^{n-i+1} - U_{k-i+1}) \right\}$$

$$+ \sum_{m=2}^{[\frac{n}{2}]} \frac{1}{m!} \sum_{i=m}^{n-m} D^m \boldsymbol{f}^i \sum_{\substack{l_1+l_2+\cdots+l_m = n-(i-m) \\ 2 \le l_1, \, l_2, \cdots, l_m \le n-(i-m)-2(m-1)}} Y_{l_1} Y_{l_2} \cdots Y_{l_m}. \qquad (8)$$

Then setting $W - U_n = 0$ results in a set of algebraic equations for solving the coefficients of the nth order SNF and the associated nonlinear transformation. The coefficients of the lower order nonlinear transformations which have not been determined in the previous steps are used to eliminate the coefficients of the nth order CNF, p_{jkl}, as many as possible and therefore, the SNF up to nth order is obtained.

The iterative formula given in equation (8) for W has been obtained and proved in the reference [20]. This formula provides an equation which involves the nth-order terms only, and thus greatly saves computational time and computer memory.

3 The SNF for a triple zero eigenvalue

Consider the following system

$$\dot{\boldsymbol{z}} = J\,\boldsymbol{z} + \boldsymbol{f}(\boldsymbol{z}), \quad \boldsymbol{z} \in \boldsymbol{R}^M, \tag{9}$$

which vanishes at the origin $\mathbf{0}$, and J is given by

$$J = \begin{bmatrix} J_1 & \\ & A \end{bmatrix} \quad \text{or} \quad J = \begin{bmatrix} J_2 & \\ & A \end{bmatrix}, \tag{10}$$

where

$$J_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad J_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{11}$$

and $A \in R^{(M-3)\times(M-3)}$ is hyperbolic, i.e., all eigenvalues of A have non-zero real parts. J_1 is said to be index one while J_2 index two.

3.1 The SNF for a triple zero of index one

By using the normal form theory (e.g., see [1,18,19]), one may find the CNF for the triple zero singularity with index one:

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= x_3, \\ \dot{x}_3 &= \sum_{k=2}^{\infty} \left\{ \sum_{i=0}^{k} a_{(k-i)\,i}\; x_1^{k-i} x_2^{i} + x_1^{I}\, x_3 \sum_{i=0}^{J} b_{N\,(J-i)\,i}\; x_1^{J-i} x_3^{i} \right\}, \end{aligned} \tag{12}$$

where

$$\begin{aligned} &N = 1, \quad J = \tfrac{1}{2}(k-1), \quad I = J, \qquad \text{when } k \text{ is odd}, \\ &N = 2, \quad J = \tfrac{k}{2} - 1, \quad I = J + 1, \qquad \text{when } k \text{ is even}. \end{aligned} \tag{13}$$

Next, the approach described in the previous section can be used to compute the SNF. We start from the second order terms (i.e., $n = 2$), and can write Y_2, V_2 and U_2, with the aid of the linear part $\boldsymbol{v_1} = (x_1,\, x_2,\, 0)^T$ as follows:

$$\begin{aligned} Y_2 &= \left(\sum_{j+k+l=2} d_{1jkl}\, x_1^j x_2^k x_3^l\,,\; \sum_{j+k+l=2} d_{2jkl}\, x_1^j x_2^k x_3^l\,,\; \sum_{j+k+l=2} d_{3jkl}\, x_1^j x_2^k x_3^l \right)^T, \\ V_2 &= \Big(0,\; 0,\; a_{20}\, x_1^2 + a_{11}\, x_1 x_2 + a_{02}\, x_2^2 + b_{200}\, x_1 x_3\Big)^T, \\ U_2 &= \boldsymbol{v_1} + \Big(0,\; 0,\; A_{20}\, x_1^2 + A_{11}\, x_1 x_2 + A_{02}\, x_2^2 + B_{200}\, x_1 x_3\Big)^T. \end{aligned} \tag{14}$$

Then solving the linear algebraic equation $V_2 + [\,Y_2,\, \boldsymbol{v_1}\,] - U_2 = 0$ yields

$$\begin{aligned}
&d_{1200} = d_{1110} = d_{2200} = d_{2020} = d_{2110} = d_{2101} = 0, \\
&d_{3200} = d_{3020} = d_{3110} = d_{3011} = d_{3101} = 0, \\
&d_{1011} = d_{2002}, \quad d_{1101} = d_{3002} - 2\, d_{1020}, \quad d_{2011} = d_{3002}, \\
&A_{20} = a_{20}, \quad A_{11} = a_{11}, \quad A_{02} = a_{02}\,, \quad B_{200} = b_{200}.
\end{aligned} \tag{15}$$

Equation (15) indicates that the second order terms of the CNF cannot be further simplified. Also, it is noted that the nonlinear coefficients d_{1020}, d_{2002}, d_{3002} and d_{1002} (d_{1002} does not appear in the algebraic equation) are not determined at this step, and therefore may be used in higher order equations to eliminate higher order CNF coefficients.

Next, consider $n = 3$, following the above step we may find the *key* linear algebraic equations:

$$\begin{pmatrix} A_{30}-a_{30} \\ A_{21}-a_{21} \\ A_{12}-a_{12} \\ A_{03}-a_{03} \\ B_{110}-b_{110} \\ B_{101}-b_{101} \end{pmatrix} + \begin{bmatrix} 0&0&0&0&0 \\ 0&3&0&0&0 \\ 0&0&2&0&0 \\ 0&0&0&1&0 \\ 0&0&1&0&0 \\ 0&0&0&0&1 \end{bmatrix} \begin{pmatrix} 0 \\ d_{3300} \\ d_{3210} \\ d_{3120} \\ d_{3111} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -2a_{20} \\ -a_{11} \\ 4\,a_{20} \\ 2\,b_{200} \end{pmatrix} d_{1020} - \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ a_{11} \end{pmatrix} d_{2002} - \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 2a_{20} \end{pmatrix} d_{1002} = \mathbf{0}$$

where the vector $(0,\ d_{3300},\ d_{3210},\ d_{3120},\ d_{3111})^T$ can be expressed in terms of the second order undetermined coefficients as follows:

$$\begin{pmatrix} d_{3300} \\ d_{3210} \\ d_{3120} \\ d_{3111} \end{pmatrix} = \begin{bmatrix} 0&0&0&0 \\ *&*&0&0 \\ *&*&*&0 \\ *&*&*&* \end{bmatrix} \begin{pmatrix} d_{3002} \\ d_{1020} \\ d_{2002} \\ d_{1002} \end{pmatrix}, \tag{16}$$

where an $*$ represents a known expression. Substituting equation (16) into the *key* equation results in

$$\begin{pmatrix} A_{30}-a_{30} \\ A_{21}-a_{21} \\ A_{12}-a_{12} \\ A_{03}-a_{03} \\ B_{110}-b_{110} \\ B_{101}-b_{101} \end{pmatrix} + \begin{bmatrix} 0&0&0&0 \\ 0&0&0&0 \\ *&*&0&0 \\ *&*&*&0 \\ *&*&0&0 \\ *&*&*&0 \end{bmatrix} \begin{pmatrix} d_{3002} \\ d_{1020} \\ d_{2002} \\ d_{1002} \end{pmatrix} = \mathbf{0}. \tag{17}$$

It is easy to see from equation (17) that $A_{30} = a_{30}$, $A_{21} = a_{21}$, and d_{1002} is not involved in the equation. Further note that the maximum rank of the matrix in equation (17) is three, and thus three of the remaining four coefficients A_{12}, A_{03}, B_{110} and B_{101} can be eliminated by appropriately choosing values for d_{3002}, d_{1020} and d_{2002}. In this paper we choose $A_{12} = A_{03} = B_{101} = 0$, and then B_{110} is uniquely determined. Of course, there exist other choices, implying that the SNF is unique only if a "form" is fixed. Therefore, *three* third order coefficients in the CNF have been removed.

The above procedure can be carried out to higher order equations. One can similarly determines the key equations, and then uses the lower order coefficients to eliminate the CNF coefficients as many as possible. The detailed process is omitted here but we list the SNF coefficients up to 8th order in Table 1, where the NT coefficients are the coefficients of lower order nonlinear transformations which are used to remove the higher order CNF coefficients.

Table 1. The SNF coefficients for a triple zero of index one.

Order	SNF coefficients	NT coefficients
2	A_{20}, A_{11}, A_{02}, B_{200}	None
3	A_{30}, A_{21}, B_{110}	d_{3002}, d_{1020}, d_{2002}
4	A_{40}, B_{210}	d_{1002}, d_{1120}, d_{1030}, d_{3003}, d_{1021}
5	A_{50}, A_{41}, B_{120}	d_{2003}, d_{1003}, d_{1040}, d_{3022}, d_{2022}, d_{1022}
6	A_{60}, A_{51}, A_{42}, B_{220}	d_{3004}, d_{2004}, d_{1140}, d_{1004}, d_{1050}, d_{1041}
7	A_{70}, A_{61}, A_{52}, B_{130}, B_{121}	d_{3023}, d_{2023}, d_{3005}, d_{1023}, d_{2005}, d_{1005}, d_{1060}
8	A_{80}, A_{71}, A_{62}, B_{230}	d_{3042}, d_{2042}, d_{1042}, d_{3024}, d_{2024} d_{1160}, d_{1024}, d_{1070}, d_{3006}

The SNF for a triple zero of index one up to 8th order is given below:

$$\begin{aligned}
\dot{u}_1 &= u_2,\\
\dot{u}_2 &= u_3,\\
\dot{u}_3 &= a_{20}u_1^2 + a_{11}u_1u_2 + a_{02}u_2^2 + b_{200}u_1u_3 + a_{30}u_1^3 + a_{21}u_1^2u_2 + B_{110}u_1^2u_3\\
&\quad + A_{40}u_1^4 + B_{210}u_1^3u_3 + A_{50}u_1^5 + A_{41}u_1^4u_2 + B_{120}u_1^4u_3 + A_{60}u_1^6 + A_{51}u_1^5u_2\\
&\quad + A_{42}u_1^4u_2^2 + B_{220}u_1^5u_3 + A_{70}u_1^7 + A_{61}u_1^6u_2 + A_{52}u_1^5u_2^2 + B_{130}u_1^6u_3\\
&\quad + B_{121}u_1^5u_3^2 + A_{80}u_1^8 + A_{71}u_1^7u_2 + A_{62}u_1^6u_2^2 + B_{230}u_1^7u_3,
\end{aligned} \tag{18}$$

where

$$\begin{aligned}
B_{110} &= -\,(a_{12}\,b_{200} - 5\,b_{200}\,b_{110} + 4\,b_{110}\,a_{02} + 8\,b_{101}\,a_{20})/(5\,b_{200} - 4\,a_{02}),\\
A_{40} &= -\,\tfrac{1}{6}(-115\,b_{200}a_{03}a_{20} - 30\,b_{200}a_{40} + 11\,b_{200}a_{11}a_{12} + 24\,a_{40}a_{02}\\
&\qquad + 92\,a_{03}a_{20}a_{02} - 42\,a_{11}b_{101}a_{20} - 13\,a_{11}a_{12}a_{02})/(5\,b_{200} - 4\,a_{02}),
\end{aligned}$$

$$
\begin{aligned}
B_{210} = -\tfrac{1}{372}(&1259\, b_{200}^2 a_{20} a_{11} a_{12} - 13690\, b_{200}^2 a_{03} a_{20}^2 - 2692\, b_{200} a_{20} a_{11} a_{12} a_{02} \\
&- 330\, b_{200} a_{11}^2 a_{03} a_{20} + 39\, b_{200} a_{11}^3 a_{12} - 4728\, b_{200} a_{11} b_{101} a_{20}^2 \\
&- 780\, b_{200} a_{30} a_{20} a_{12} - 90\, a_{11} b_{200} a_{31} a_{20} + 23312\, b_{200} a_{03} a_{20}^2 a_{02} \\
&- 1860\, b_{200} b_{210} a_{20}^2 + 600\, b_{200} a_{22} a_{20}^2 + 1416\, a_{20} a_{11} a_{12} a_{02}^2 \\
&+ 3120\, a_{30} a_{20}^2 b_{101} + 264\, a_{11}^2 a_{03} a_{20} a_{02} + 936\, a_{30} a_{20} a_{12} a_{02} \\
&+ 4464\, a_{11} b_{101} a_{20}^2 a_{02} - 42\, a_{11}^3 a_{12} a_{02} + 1488\, b_{210} a_{20}^2 a_{02} \\
&+ 72\, a_{11} a_{31} a_{20} a_{02} - 108\, a_{11}^3 b_{101} a_{20} - 9888\, a_{03} a_{20}^2 a_{02}^2 \\
&- 480\, a_{22} a_{20}^2 a_{02})/a_{20}^2\,(5b_{200} - 4a_{02}), \\
\vdots& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (19)
\end{aligned}
$$

3.2 *The SNF for a triple zero of index two*

By using normal form theory we may find the CNF for the case of index two:

$$
\begin{aligned}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= \sum_{k=2}^{\infty} \left\{ \sum_{i=0}^{k} a_{k-i,i}\; x_1^{k-i} x_3^i + x_2 \sum_{i=0}^{k-1} b_{k-i-1,i}\; x_1^{k-i-1} x_3^i \right\}, \\
\dot{x}_3 &= \sum_{k=2}^{\infty} \sum_{i=0}^{k} c_{k-i,i}\; x_1^{k-i} x_3^i. \qquad (20)
\end{aligned}
$$

Similarly, we may follow the procedure used in obtaining the SNF for the case of index one to find the SNF for this case. First consider $n = 2$ and define

$$
\begin{aligned}
Y_2 &= \left(\sum_{j+k+l=2} d_{1jkl}\, x_1^j x_2^k x_3^l\,,\; \sum_{j+k+l=2} d_{2jkl}\, x_1^j x_2^k x_3^l\,,\; \sum_{j+k+l=2} d_{3jkl}\, x_1^j x_2^k x_3^l \right)^T, \\
V_2 &= \Big(0,\; a_{20}\, x_1^2 + a_{11}\, x_1 x_3 + a_{02}\, x_3^2 + b_{10}\, x_1 x_2 + b_{01}\, x_2 x_3, \\
&\qquad c_{20}\, x_1^2 + c_{11}\, x_1 x_3 + c_{02}\, x_3^2\Big)^T, \\
U_2 &= \boldsymbol{v_1} + \Big(0,\; A_{20}\, x_1^2 + A_{11}\, x_1 x_3 + A_{02}\, x_3^2 + B_{10}\, x_1 x_2 + B_{01}\, x_2 x_3, \\
&\qquad C_{20}\, x_1^2 + C_{11}\, x_1 x_3 + C_{02}\, x_3^2\Big)^T, \qquad (21)
\end{aligned}
$$

where $\boldsymbol{v_1} = (x_1,\, 0,\, 0)^T$ represents the linear part. Then one may solve the linear equation given by $V_2 + [\,Y_2,\, \boldsymbol{v}_1\,] - U_2 = 0$ to find

$$
\begin{aligned}
&d_{1200} = d_{2200} = d_{2002} = d_{2101} = d_{2110} = d_{3200} = d_{3110} = d_{3101} = 0, \\
&d_{1110} = d_{2020},\quad d_{2011} = d_{1101},\quad A_{20} = a_{20},\quad A_{11} = a_{11},\quad A_{02} = a_{02}, \\
&B_{10} = b_{10},\quad B_{01} = b_{01},\quad C_{20} = c_{20},\quad C_{11} = c_{11},\quad C_{02} = c_{02}, \qquad (22)
\end{aligned}
$$

which implies that the second order terms of the CNF cannot be further simplified. It is noted that the nonlinear coefficients d_{1020}, d_{1002}, d_{1110}, d_{1101}, d_{1011}, d_{3020}, d_{3002} and d_{3011} are not determined at this step, and may be used to eliminate higher order CNF coefficients.

Next, consider $n = 3$, by following the similar procedure we may find the *key* linear algebraic equations as follows:

$$\begin{pmatrix} B_{20}-b_{20} \\ B_{11}-b_{11} \\ B_{02}-b_{02} \end{pmatrix} + \begin{bmatrix} 3\,a_{20} & 0 & 0 & 4\,c_{20} & 0 \\ 3\,a_{11} & 4\,c_{20} & -\,a_{11} & 3c_{11}-b_{10} & 0 \\ 3\,a_{02} & 2\,c_{11}-b_{10} & -\,2\,a_{02} & 2\,c_{02} & -\,b_{01} \end{bmatrix} \begin{pmatrix} d_{2020} \\ d_{1002} \\ d_{3011} \\ d_{1101} \\ d_{3002} \end{pmatrix} = \mathbf{0} \tag{23}$$

and

$$\begin{pmatrix} A_{30}-a_{30} \\ A_{21}-a_{21} \\ A_{12}-a_{12} \\ A_{03}-a_{03} \\ C_{30}-c_{30} \\ C_{21}-c_{21} \\ C_{12}-c_{12} \\ C_{03}-c_{03} \end{pmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\,a_{20} & 0 \\ 0 & -\,2\,a_{20} & 0 & 0 & -\,a_{11} \\ 0 & -\,a_{11} & 0 & a_{02} & -\,2\,a_{02} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{20} & -\,2\,c_{20} & 2\,c_{20} \\ 0 & -\,2\,c_{20} & a_{11} & -\,c_{11} & c_{11} \\ 0 & -\,c_{11} & a_{02} & 0 & 0 \end{bmatrix} \begin{pmatrix} d_{2020} \\ d_{1002} \\ d_{3011} \\ d_{1101} \\ d_{3002} \end{pmatrix} = \mathbf{0}. \tag{24}$$

It is easy to see from equation (23) that we can use three of the d's coefficients to eliminate the three normal form coefficients B_{20}, B_{11} and B_{02}, which further simplifies the CNF. For definite, here we choose d_{2020}, d_{1002} and d_{3011}, and solve them from equation (23) (noting that $B_{20} = B_{11} = B_{02} = 0$), and then substitute the resulting solutions into equation (24) to further eliminate two of the coefficients A's and/or C's by using the coefficients d_{1101} and d_{3002}. However, it should be pointed that the coefficients A_{30} and C_{30} cannot be set zero, and in fact, $A_{30} = a_{30}$ and $C_{30} = c_{30}$. Here we may choose $A_{21} = A_{12} = 0$ and then uniquely determine d_{1101} and d_{3002} from equation (24). It has also been noted that the coefficient d_{1011} does not appear in equations (23) and (24), and may be used later to remove some higher order CNF coefficient.

The above procedure can be repeatly applied to higher order equations: First determines the *key equations*, and then uses the lower order undetermined coefficients to eliminate the higher order CNF coefficients as many as possible. Table 2 shows the SNF coefficients up to 8th order and the nonlinear transformation (NT) coefficients which are the lower order coefficients used to eliminate the higher order CNF coefficients.

Table 2. The SNF coefficients for a triple zero of index two.

Order	SNF coefficients	NT coefficients
2	A_{20}, A_{11}, A_{02}, B_{10}, B_{01} C_{20}, C_{11}, C_{02}	None
3	A_{30}, A_{03} C_{30}, C_{21}, C_{12}, C_{03}	d_{1101}, d_{1002}, d_{2020}, d_{3002}, d_{3011}
4	A_{40} C_{40}, C_{31}, C_{22}, C_{13}, C_{04}	d_{1102}, d_{1011}, d_{1111}, d_{1003} d_{1020}, d_{3020}, d_{3012}, d_{3003}
5	A_{50} C_{41}, C_{32}, C_{23}, C_{14}, C_{05}	d_{1004}, d_{1012}, d_{1103}, d_{1021}, d_{1120}, d_{1112} d_{3021}, d_{3004}, d_{3013}, d_{3112}, d_{3030}
6	A_{60}, C_{60}, C_{51} C_{42}, C_{33}, C_{24}, C_{15}, C_{06}	d_{1113}, d_{1104}, d_{1121}, d_{1013}, d_{1022}, d_{1005} d_{1030}, d_{1130}, d_{3022}, d_{3031}, d_{3014}, d_{3005}
7	A_{70}, A_{43}, A_{34}, A_{25}, A_{16} C_{70}, C_{61}, C_{34}, C_{25}	d_{1040}, d_{1122}, d_{1023}, d_{1014}, d_{1114}, d_{1105}, d_{1031} d_{1131}, d_{1006}, d_{3006}, d_{3015}, d_{3023}, d_{3032}, d_{3040}
8	A_{80}, C_{80}, C_{71}, C_{62}, C_{53} C_{44}, C_{35}, C_{26}, C_{17}, C_{08}	d_{1007}, d_{1115}, d_{1106}, d_{3016}, d_{3007} d_{1123}, d_{1015}, d_{1024}, d_{1132}, d_{3024}, d_{3033} d_{1041}, d_{1032}, d_{1140}, d_{3041}, d_{3050}

Finally, the SNF for a triple zero of index two up to 8th order is found to be:

$$
\begin{aligned}
\dot{u}_1 &= u_2, \\
\dot{u}_2 &= a_{20}u_1^2 + a_{11}u_1u_3 + a_{02}u_3^2 + b_{10}u_1u_2 + b_{01}u_2u_3 + A_{30}u_1^3 + A_{03}u_3^3 + A_{40}u_1^4 \\
&\quad + A_{50}u_1^5 + A_{06}u_3^6 + A_{70}u_1^7 + A_{80}u_1^8, \\
\dot{u}_3 &= C_{20}u_1^2 + C_{11}u_1u_3 + C_{02}u_3^2 + C_{30}u_1^3 + C_{21}u_1^2u_3 + C_{12}u_1u_3^2 + C_{03}u_3^3 \\
&\quad + C_{40}u_1^4 + C_{31}u_1^3u_3 + C_{22}u_1^2u_3^2 + C_{13}u_1u_3^3 + C_{04}u_3^4 + C_{41}u_1^4u_3 \\
&\quad + C_{32}u_1^3u_3^2 + C_{23}u_1^2u_3^3 + C_{14}u_1u_3^4 + C_{05}u_3^5 + C_{60}u_1^6 + C_{51}u_1^5u_3 \\
&\quad + C_{42}u_1^4u_3^2 + C_{33}u_1^3u_3^3 + C_{24}u_1^2u_3^4 + C_{15}u_1u_3^5 + C_{06}u_3^6 + C_{70}u_1^7 \\
&\quad + C_{61}u_1^6u_3 + C_{34}u_1^3u_3^4 + C_{25}u_1^2u_3^5 + C_{80}u_1^8 + C_{71}u_1^7u_3 + C_{62}u_1^6u_3^2 \\
&\quad + C_{53}u_1^5u_3^3 + C_{44}u_1^4u_3^4 + C_{35}u_1^3u_3^5 + C_{26}u_1^2u_3^6 + C_{17}u_1u_3^7 + C_{08}u_3^8, \qquad (25)
\end{aligned}
$$

where the coefficients A_{jk}'s and C_{jk}'s are given in terms of the known coefficients a_{jk}'s, b_{jk}'s and c_{jk}'s. The detailed expressions are omitted here.

It is noted from equation (11) that the Jacobian J_2 involves a non-semisimple double zero eigenvalue. The explicit SNF for the double zero eigenvalue has been obtained in [16]. Here we may use the SNF for the triple zero of index two, given by equation (25), to recover the SNF of the generic case for the double zero singularity. First note in equation (20) that setting all c coefficients zero in the 3rd equation of (20), and taking the terms in the second equation of (20) for $i = 0$ only results in the CNF for the double zero singularity. This leads to the coefficients of equation (25): $C_{20} = C_{30} = C_{40} = C_{60} = C_{70} = C_{80} = 0$. Then further setting $u_3 = 0$ in equation (25) yields

$$\begin{aligned} \dot{u}_1 &= u_2\,, \\ \dot{u}_2 &= a_{20}u_1^2 + b_{10}u_1u_2 + A_{30}u_1^3 + A_{40}u_1^4 + A_{50}u_1^5 + A_{70}u_1^7 + A_{80}u_1^8. \end{aligned} \tag{26}$$

A careful examination of the SNF for the double zero singularity has shown that the result given by equation (26) is exactly same as that obtained in [16].

3.3 Symbolic computation using Maple

The explicit formulas derived in this section can be directly implemented on computer algebra systems. In fact, all the results obtained in this paper are obtained using Maple. The Maple programs we have developed can be conveniently executed on any computer systems for computing the SNF of a given vector field associated with the triple zero singularity of index one or two. The software only requires a minimum preparation for an input file from a user.

4 Conclusions

A method is presented for computing the simplest normal forms of differential equations associated with triple zero singularity of index one and two. It has been shown that this approach is an efficient way from the computation point of view. User-friendly symbolic computation programs written in Maple have been developed. This method can be easily extended to consider other singularities.

References

[1] Guckenheimer, J., & and Holmes, P., *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, Springer-Verlag (New York, 1983).

[2] Golubisky, M.S., & Schaeffer, D.G., *Singularities and Groups in Bifurcation Theory*, Springer-Verlag (New York, 1985).

[3] Chow, S.-N., Li, C., & and Wang, D., *Normal Forms and Bifurcation of Planar Vector Fields*, Cambridge University Press (Cambridge, 1994).

[4] Ushiki, S., Normal forms for singularities of vector fields, 1984, Japan J. Appl. Math., 1, 1–37.

[5] Baider, A., & Churchill, R., Unique normal forms for planar vector fields, 1988, Math. Z., 199, 303-310.

[6] Yu, P., Simplest normal forms of Hopf and generalized Hopf bifurcations, 1999, Int. J. Bifurcation and Chaos, 9, 1917-1939.

[7] Baider, A., & Sanders, J.A., Further reduction of the Takens-Bogdanov normal forms, 1992, J. Diff. Eqn., 99, 205–244.

[8] Kokubu, H., Oka, H., & Wang, D., Linear grading function and further reduction of normal forms, 1996, J. Diff. Eqn., 132, 293–318.

[9] Wang, D., Li, J., Huang, M., & Jiang, Y., Unique normal form of Bogdanov-Takens Singularities, 2000, J. Diff. Eqn., 163, 223-238.

Takens Singularities, 2000, J. Diff. Eqn., 163, 223-238