Ribbon transformations, Integrals, and Triangular Decompositions

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Abstract

We study the theory of integrals in Yetter-Drinfel'd Hopf algebras and use the results to determine the integrals of Hopf algebras with triangular decomposition.

1 Introduction

The question whether it is possible to understand the algebra structure of deformed enveloping algebras in terms of the algebra structure of the three subalgebras appearing in the triangular decomposition was addressed in [31] (cf. also [32]). For this purpose, one can use two Hopf-algebraic constructions, the first one of these two being a very general one which might describe the structure of other Hopf algebras, too. In these constructions, two of the three components are not ordinary Hopf algebras but Hopf algebras in a certain twisted sense, that is, Hopf algebras in the category of Yetter-Drinfel'd modules. It is the purpose of this paper to determine the integrals of Hopf algebras that admit a triangular decomposition which is similar to the triangular decomposition of deformed enveloping algebras. For this purpose, we carry out a comprehensive investigation of the properties of integrals of Yetter-Drinfel'd Hopf algebras.

The article is organized as follows: In section 2, we recall some results of D. Fischman, S. Montgomery and H.-J. Schneider (cf. [5]) concerning the existence and uniqueness of integrals in Yetter-Drinfel'd Hopf algebras, as well as the definition of the integral character and the integral group element. It must be emphasized that these authors have established these results even under much more general hypotheses. In contrast to their methods, the proofs in section 2 are direct analogues to the proofs for ordinary Hopf algebras. These proofs also carry over directly to more general quasisymmetric categories as already observed earlier by V. Lyubashenko (cf. [16],[17]). As H.-J. Schneider has pointed out, these facts are also recalled in a recent preprint of Y. Doi (cf. [4]). It should be noted that important parts of these results have already been shown much earlier by D. Radford (cf. [23]). After this repetition, we study the properties of the modular functions and elements as well as the properties of integral character and the integral group element. In particular, we show that these are central elements in a certain sense.

In section 3, we lay the abstract foundations that make it easier to understand the nature of certain maps that arise naturally when considering Yetter-Drinfel'd Hopf algebras. We recall the notion of a monoidal transformation and contrast it with the related notion of a ribbon transformation. We exhibit examples for monoidal and ribbon transformations in the category of Yetter-Drinfel'd modules, and explain the effect of the action of such transformations on Yetter-Drinfel'd Hopf algebras. Two kinds of monoidal transformations, the modular transformations and the integral transformation, play the essential role in section 4 where we introduce the twisted Nakayama automorphisms and prove an analogue of Radford's formula for the fourth power of the antipode of a Hopf algebra in the Yetter-Drinfel'd case.

In section 5, we determine the integrals of Hopf algebras with triangular decomposition that arise from the above mentioned constructions. It turns out to be difficult to determine the integrals for the first construction, while the integrals of the second construction can be easily obtained afterwards. These results contain as a special case the formula for the integrals of the Drinfel'd double construction obtained by D. Radford (cf. [24]). In the last section 6, we illustrate the theory by considering the example of the Frobenius-Lusztig kernel of $sl(2)$. The Frobenius-Lusztig kernels of deformed enveloping algebras were defined by G. Lusztig (cf. [14], [15]). Their present name was introduced by N. Andruskiewitsch and H. J. Schneider (cf. [1]).

In this article, all vector spaces occurring are defined over a base field K. An excellent general reference for all topics discussed here is [19].

2 Integrals and Yetter-Drinfel'd modules

2.1 In this preliminary section we give a self-contained exposition of some results of D. Fischman, S. Montgomery and H.-J. Schneider (cf. [5]). Some of these results were proved earlier by D. Radford (cf. [23]). We work in the following situation: H denotes a Hopf algebra with bijective antipode defined over the field K having the comultiplication Δ_H and the counit ϵ_H. We use the following Heyneman-Sweedler sigma notation:

$$\Delta_H(h) = h_1 \otimes h_2$$

Recall the notion of a left Yetter-Drinfel'd module (cf. [38], [19], Def. 10.6.10): This is a left H-comodule V which is also a left H-module such that the following compatibility condition is satisfied:

$$h_1 v^1 \otimes (h_2 \rightharpoonup v^2) = (h_1 \rightharpoonup v)^1 h_2 \otimes (h_1 \rightharpoonup v)^2$$

for all $h \in H$ and $v \in V$. Here we have used the following Sweedler notation for the coaction: $\delta(v) = v^1 \otimes v^2 \in H \otimes V$. The arrow $\rightharpoonup$ denotes the module action. We also define right Yetter-Drinfel'd modules, which are the left Yetter-Drinfel'd modules over the opposite and coopposite Hopf algebra. They are right comodules and right modules that satisfy:

$$(v^1 \leftharpoonup h_1) \otimes v^2 h_2 = (v \leftharpoonup h_2)^1 \otimes h_1 (v \leftharpoonup h_2)^2$$

Of course one can also define left-right and right-left Yetter-Drinfel'd modules, which are the left Yetter-Drinfel'd modules over the opposite resp. coopposite Hopf algebra, but they are not used in this article.

2.2 The tensor product of two Yetter-Drinfel'd modules becomes again a Yetter-Drinfel'd module if it is endowed with the diagonal module and the codiagonal comodule structure (cf. [19], Example 10.6.14). The base field K becomes a Yetter-Drinfel'd module via the trivial module structure $h \rightharpoonup \xi := \epsilon_H(h)\xi$ and the trivial comodule structure $\delta_K(\xi) := 1 \otimes \xi$. The left Yetter-Drinfel'd modules, and also the right ones, therefore constitute a monoidal category. But these categories also possess braidings, which are in the left case given by:

$$\sigma_{V,W} : V \otimes W \longrightarrow W \otimes V$$
$$v \otimes w \mapsto (v^1 \rightharpoonup w) \otimes v^2$$

The corresponding formula in the right case reads:

$$\sigma_{V,W}(v \otimes w) = w^1 \otimes (v \leftharpoonup w^2).$$

These mappings are bijective since H has a bijective antipode.

Since we have the notion of a Hopf algebra inside a braided monoidal category (cf. [28]), we can speak of Yetter-Drinfel'd Hopf algebras.

2.3 There are several elementary operations with Yetter-Drinfel'd modules which will be needed in the sequel.

Lemma If V is a finite dimensional left Yetter-Drinfel'd module, then the dual space V^* is in a unique way a right Yetter-Drinfel'd module such that the natural pairing

$$\langle \cdot, \cdot \rangle : V \times V^* \to K, (v, f) \mapsto f(v)$$

is a Yetter-Drinfel'd form (cf. [32], subsection 2.4), i. e. that we have

1. $\langle h \rightharpoonup v, f \rangle = \langle v, f \leftharpoonup h \rangle$
2. $\langle v, f^1 \rangle f^2 = v^1 \langle v^2, f \rangle$

Proof. An element of H acts on V^* via the transpose of the action on V. The comodule-structure is given by the formula:

$$\delta_{V^*}(f) = \sum_{i=1}^{n} v^{(i)*} \otimes f({v_{(i)}}^2){v_{(i)}}^1$$

where $v_{(1)}, \dots, v_{(n)}$ is a basis of V with dual basis $v^{(1)*}, \dots, v^{(n)*}$. □

The transpose of an H-linear and colinear map between finite-dimensional Yetter-Drinfel'd modules is linear and colinear. If V_1 and V_2 are finite-dimensional left Yetter-Drinfel'd modules, then ${V_1}^* \otimes {V_2}^*$ is isomorphic to $(V_1 \otimes V_2)^*$ as a right Yetter-Drinfel'd module via the isomorphism

$${V_1}^* \otimes {V_2}^* \to (V_1 \otimes V_2)^*, f_1 \otimes f_2 \mapsto (v_1 \otimes v_2 \mapsto f_1(v_1)f_2(v_2))$$

Up to this isomorphism, the braiding on ${V_1}^* \otimes {V_2}^*$ is the transpose of the braiding on $V_1 \otimes V_2$. One can express these facts by saying that taking the dual space is a (non-strict) braided monoidal functor from the category of finite-dimensional left Yetter-Drinfel'd modules to the category of finite-dimensional right Yetter-Drinfel'd modules (cf. [9], Definition 2.3). As a consequence, if A is a finite-dimensional left Yetter-Drinfel'd Hopf algebra, then the dual space $B := A^*$ is in a unique way a right Yetter-Drinfel'd Hopf algebra such that the natural pairing described above is a bialgebra form (cf. [32], subsection 2.6), i. e. we have:

1. $\langle a \otimes a', \Delta_B(b) \rangle = \langle aa', b \rangle$
2. $\langle a, bb' \rangle = \langle \Delta_A(a), b \otimes b' \rangle$
3. $\langle 1, b \rangle = \epsilon_B(b)$, $\langle a, 1 \rangle = \epsilon_A(a)$

2.4 If H is finite-dimensional, the process of dualization can also be applied to H:

Lemma Suppose that H is finite dimensional. If V is a left Yetter-Drinfel'd module over H, then V becomes a right Yetter-Drinfel'd module over H^* using the module structure

$$v \leftarrow f := f(v^1)v^2$$

for $v \in V$ and $f \in V^*$, and the comodule structure

$$\delta_*(v) := \sum_{i=1}^{n} (h_{(i)} \to v) \otimes h^{(i)*}$$

where $h_{(1)}, \dots, h_{(n)}$ is a basis of H with dual basis $h^{(1)*}, \dots, h^{(n)*}$.

Proof. This rests on direct computation (cf. [19], Lemma 1.6.4). □

If H is finite dimensional, a mapping between two Yetter-Drinfel'd modules is linear and colinear with respect to H if and only if it is linear and colinear with respect to H^*. The dualization process described in this Lemma commutes with taking the tensor product of two Yetter-Drinfel'd modules, and the braiding on the tensor product is the same before and after the dualization. This can be expressed by saying that dualization with respect to H gives rise to a strict braided monoidal functor from the category of left Yetter-Drinfel'd modules over H to right Yetter-Drinfel'd modules over H^*. Therefore, a Hopf algebra in the former category remains a Hopf algebra in the latter category.

2.5 Since left Yetter-Drinfel'd modules are the same as right Yetter-Drinfel'd modules over the opposite and coopposite Hopf algebra, we have:

Lemma

1. If A is a left Yetter-Drinfel'd Hopf algebra over H, then the opposite and coopposite Hopf algebra $A^{op\ cop}$ is a right Yetter-Drinfel'd Hopf algebra over $H^{op\ cop}$.
2. If B is a right Yetter-Drinfel'd Hopf algebra over H, then $B^{op\ cop}$ is a left Yetter-Drinfel'd Hopf algebra over $H^{op\ cop}$.

The proof is omitted.

2.6 We now proceed to explain a slightly more interesting operation that can be carried out with Yetter-Drinfel'd Hopf algebras. In this article, a character is an algebra map to the base field. A character is the same as an augmentation. A character in the sense used here is a one-dimensional character in the more general sense of the word used in representation theory.

Proposition Suppose that A is a left Yetter-Drinfel'd Hopf algebra over H and that $\omega : A \rightarrow K$ is an H-linear and colinear character. We have:

1. A can be regarded in another way as a left Yetter-Drinfel'd Hopf algebra if the comultiplication is changed to
$$\Delta_A^\omega(a) := a_1 \otimes \omega(a_2)a_3$$
and counit and antipode are changed to
$$\epsilon_A^\omega(a) := \omega^{-1}(a), \qquad S_A^\omega(a) := \omega^{-1}(a_1)S_A(a_2)\omega^{-1}(a_3).$$
Here we have adopted the notation $\omega^{-1}(a) := \omega(S_A(a))$. We shall denote A by A^ω if it is endowed with this structure.
2. The mappings
$$\phi : A \rightarrow A^\omega, a \mapsto \omega(a_1)a_2$$
$$\psi : A \rightarrow A^\omega, a \mapsto a_1\omega(a_2)$$
are isomorphisms of Yetter-Drinfel'd Hopf algebras.

Proof. It is of course sufficient to prove the second assertion, because then the structure in 1. will appear as the structure of A carried over to somewhere else via a bijective map. Since up to the identification of $K \otimes A$ resp. $A \otimes K$ with A, which is H-linear and colinear, we have:

$$\phi = (\omega \otimes id_A) \circ \Delta_A \qquad \psi = (id_A \otimes \omega) \circ \Delta_A,$$

we see that ϕ and ψ are linear and colinear as compositions of linear and colinear maps. They are obviously bijective with inverses:

$$\phi^{-1}(a) = \omega^{-1}(a_1)a_2 \qquad \psi^{-1}(a) = a_1\omega^{-1}(a_2)$$

ϕ is an algebra map since ω is linear:

$$\begin{aligned}\phi(aa') &= \omega(a_1({a_2}^1 \rightarrow a'_1)){a_2}^2 a'_2 \\ &= \omega(a_1)\omega(a'_1)a_2a'_2 = \phi(a)\phi(a')\end{aligned}$$

ψ is an algebra map since ω is colinear:

$$\begin{aligned}\psi(aa') &= a_1({a_2}^1 \rightarrow a'_1)\omega({a_2}^2 a'_2) \\ &= a_1a'_1\omega(a_2)\omega(a'_2) = \psi(a)\psi(a')\end{aligned}$$

ϕ and ψ obviously preserve the unit and the counit. They are coalgebra maps because we have:

$$\begin{aligned}\Delta_A^\omega(\phi(a)) &= \omega(a_1)a_2 \otimes \omega(a_3)a_4 = (\phi \otimes \phi) \circ \Delta_A(a) \\ \Delta_A^\omega(\psi(a)) &= a_1\omega(a_2) \otimes a_3\omega(a_4) = (\psi \otimes \psi) \circ \Delta_A(a)\end{aligned}$$

Finally, we observe that ϕ and ψ commute with the antipode:

$$\begin{aligned}\phi(S_A(a)) &= \omega(S_A({a_1}^1 \rightarrow a_2))S_A({a_1}^2) \\ &= \omega(S_A(a_2))S_A(a_1) \\ &= \omega(a_1)\omega^{-1}(a_2)S_A(a_3)\omega^{-1}(a_4) \\ &= S_A^\omega(\phi(a))\end{aligned}$$

by the linearity of ω and

$$\begin{aligned}\psi(S_A(a)) &= S_A({a_1}^1 \rightarrow a_2)\omega(S_A({a_1}^2)) \\ &= S_A(a_2)\omega^{-1}(a_1) \\ &= \omega^{-1}(a_1)S_A(a_2)\omega^{-1}(a_3)\omega(a_4) \\ &= S_A^\omega(\psi(a))\end{aligned}$$

by the colinearity of ω. $\square$

This Proposition of course also holds if H is equal to the base field K; in this case Yetter-Drinfel'd Hopf algebras are ordinary Hopf algebras. We therefore see that there is no essential difference between the counit and an arbitrary character: If one character of an algebra arises as the counit of a suitable Hopf algebra structure, then every character arises in this way.

It should be observed that in proving that ϕ is an algebra map only the linearity of ω was needed, whereas the fact that ψ is an algebra map follows solely from the colinearity of ω. This fact will be needed in the sequel.

2.7 Recall that a Frobenius algebra is a finite-dimensional algebra A which admits a nondegenerate bilinear form $\langle \cdot, \cdot \rangle : A \otimes A \to K$ which is associative in the sense that we have $\langle aa', a'' \rangle = \langle a, a'a'' \rangle$ for all a, a' and $a'' \in A$. Such a form obviously can be written as

$$\langle a, a' \rangle = f(aa')$$

for some linear form $f : A \to K$ which is determined by the bilinear form via $f(a) = \langle a, 1 \rangle = \langle 1, a \rangle$. This linear form is called the Frobenius homomorphism. We note that this notion is not related to the same term used in Galois theory.

The following Proposition was explained to me by H.-J. Schneider:

Proposition Suppose that A is a Frobenius algebra which is augmented by $\epsilon : A \to K$. Then the space of left integrals $\{x \in A \mid \forall a \in A : ax = \epsilon(a)x\}$, and also the space of right integrals $\{x \in A \mid \forall a \in A : xa = \epsilon(a)x\}$, is one-dimensional.

Proof. Consider the left coregular action of A on A^*: An element a of A acts on an element g of A^* yielding the element $a \rightharpoonup g$ of A^* which is defined as $(a \rightharpoonup g)(a') = g(a'a)$. One can restate the definition of a Frobenius algebra by saying that the mapping

$$A \to A^*, a \mapsto (a \rightharpoonup f)$$

is bijective, that is, A^* is a free cyclic A-module generated by the Frobenius homomorphism f. Our assertion will be proved if we can show that x is a left integral if and only if $x \rightharpoonup f$ is a multiple of ϵ. But observe that:

$$\begin{aligned}
\exists \lambda \in K : x \rightharpoonup f = \lambda\epsilon &\Leftrightarrow \exists \lambda \in K \; \forall a \in A : f(ax) = \lambda\epsilon(a) \\
&\Leftrightarrow \forall a \in A : f(ax) = f(x)\epsilon(a) \\
&\Leftrightarrow \forall a, a' \in A : f(aa'x) = f(x)\epsilon(aa') = f(x)\epsilon(a)\epsilon(a') \\
&\Leftrightarrow \forall a, a' \in A : f(aa'x) = f(a\epsilon(a')x) \\
&\Leftrightarrow \forall a' \in A : a'x = \epsilon(a')x
\end{aligned}$$

The assertion on right integrals follows by considering the opposite algebra A^{op} instead of A. $\square$

Since the form $\langle \cdot, \cdot \rangle$ is nondegenerate, we can choose dual bases $x_{(1)}, \dots, x_{(n)}$ and $y_{(1)}, \dots, y_{(n)}$ satisfying $\langle y_{(i)}, x_{(j)} \rangle = \delta_{ij}$. From linear algebra we know that we have $a = \sum_{i=1}^{n} \langle a, x_{(i)} \rangle y_{(i)} = \sum_{i=1}^{n} \langle y_{(i)}, a \rangle x_{(i)}$ for all $a \in A$. This implies that we have:

$$\sum_{i=1}^{n} a x_{(i)} \otimes y_{(i)} = \sum_{i=1}^{n} x_{(i)} \otimes y_{(i)} a$$

The element $\sum_{i=1}^{n} x_{(i)} \otimes y_{(i)}$ is therefore called the Casimir element of A. It does not depend on the choice of the dual bases, but of course it does depend on the bilinear form.

2.8 We now proceed to prove the existence a non-zero integral in every finite dimensional Yetter-Drinfel'd Hopf algebra strictly along the lines of the original argument of R. Larson and M. Sweedler (cf. [12], see also [19]). It must be emphasized that this result is not new, but is a special case of a Theorem of D. Fischman, S. Montgomery and H.-J. Schneider on the one hand and V. Lyubashenko on the other hand (cf. [5], [16], [17]). We include the result here for the sake of completeness and because the arguments of the proof will be needed later on.

The basic ingredient of the proof is the structure theorem for Hopf modules. Recall the notion of a Hopf module in the category of Yetter-Drinfel'd modules:

Definition Suppose that A is a left Yetter-Drinfel'd Hopf algebra over H. A right Yetter-Drinfel'd Hopf module is a left Yetter-Drinfel'd module M which is a right A-module via:

$$\phi_M : M \otimes A \rightarrow M$$

and a right A-comodule via:

$$\delta_M : M \rightarrow M \otimes A$$

such that ϕ_M and δ_M are H-linear and H-colinear and the following compatibility condition is satisfied:

$$(m \leftarrow a)^{(1)} \otimes (m \leftarrow a)^{(2)} = (m^{(1)} \leftarrow (m^{(2)1} \rightarrow a_1)) \otimes m^{(2)2} a_2$$

for all $m \in M$ and $a \in A$, where we have used the notation:

$$\phi_M(m \otimes a) = m \leftarrow a \qquad \delta_M(m) = m^{(1)} \otimes m^{(2)}$$

The coaction δ_M should be distinguished from the coaction $\delta : M \rightarrow H \otimes M$ which comes from the Yetter-Drinfel'd structure. Examples of Yetter-Drinfel'd

Hopf modules can be easily obtained in the following way: If A is a Yetter-Drinfel'd Hopf algebra and V is any Yetter-Drinfel'd module, then $M := V \otimes A$ becomes a Yetter-Drinfel'd Hopf module with respect to the structure elements:

$$\delta_M : V \otimes A \to (V \otimes A) \otimes A, v \otimes a \mapsto v \otimes a_1 \otimes a_2$$
$$\phi_M : (V \otimes A) \otimes A \to V \otimes A, v \otimes a \otimes a' \mapsto v \otimes aa'$$

The structure theorem for Hopf modules now asserts that this is already the general form of a Yetter-Drinfel'd Hopf module:

Proposition Suppose that A is a Yetter-Drinfel'd Hopf algebra over H and that M is a right Yetter-Drinfel'd Hopf module over A. Define the space of coinvariants:

$$V := M^{coA} = \{m \in M \mid \delta_M(m) = m \otimes 1\}$$

Then V is a Yetter-Drinfel'd submodule of M and

$$p : M \to V, m \mapsto (m^{(1)} \leftharpoonup S_A(m^{(2)}))$$

is a projection onto V. Furthermore, the mappings

$$f : V \otimes A \to M, v \otimes a \mapsto (v \leftharpoonup a)$$
$$g : M \to V \otimes A, m \mapsto p(m^{(1)}) \otimes m^{(2)}$$

are linear and colinear with respect to A and H and are mutually inverse to each other.

Proof. We first prove that p really maps to the coinvariants:

$$\begin{aligned}
\delta_M(p(a)) &= \delta_M(m^{(1)} \leftharpoonup S_A(m^{(2)})) \\
&= (m^{(1)} \leftharpoonup (m^{(2)1} \rightharpoonup S_A(m^{(3)})_1)) \otimes m^{(2)2} S_A(m^{(3)})_2 \\
&= (m^{(1)} \leftharpoonup (S_A(m^{(2)1} m^{(3)1} \rightharpoonup m^{(4)}))) \otimes m^{(2)2} S_A(m^{(3)2}) \\
&= (m^{(1)} \leftharpoonup (S_A(m^{(2)}{}_1{}^1 m^{(2)}{}_2{}^1 \rightharpoonup m^{(3)}))) \otimes m^{(2)}{}_1{}^2 S_A(m^{(2)}{}_2{}^2) \\
&= (m^{(1)} \leftharpoonup (S_A(m^{(2)1} \rightharpoonup m^{(3)}))) \otimes m^{(2)2}{}_1 S_A(m^{(2)2}{}_2) \\
&= (m^{(1)} \leftharpoonup (S_A(m^{(2)1} \rightharpoonup m^{(3)}))) \otimes \epsilon_A(m^{(2)2}) 1 \\
&= (m^{(1)} \leftharpoonup S_A(m^{(2)})) \otimes 1
\end{aligned}$$

We now prove that V is an H-subcomodule of M. Since we have $\delta_M(m) = m \otimes 1$ for $m \in V$, we can compute:

$$m^1 \otimes \delta_M(m^2) = m^{(1)1} m^{(2)1} \otimes m^{(1)2} \otimes m^{(2)2} = m^1 \otimes m^2 \otimes 1$$

Writing $m^1 \otimes m^2 = \sum_{b \in B} b \otimes x_b$ for some basis B of H, we see that $x_b \in M^{coA}$. In a similar fashion, it is possible to prove that V is an H-submodule of M. The Yetter-Drinfel'd condition therefore holds for V since it holds for M. Next, we observe that the mappings p, f and g are linear and colinear with respect to H because they are compositions of H-linear and colinear maps. We prove that we have $p(m \leftharpoonup a) = \epsilon_A(a)p(m)$ for $m \in M$, $a \in A$:

$$\begin{aligned} p(m \leftharpoonup a) &= (m \leftharpoonup a)^{(1)} \leftharpoonup S_A((m \leftharpoonup a)^{(2)}) \\ &= (m^{(1)} \leftharpoonup (m^{(2)1} \rightharpoonup a_1)) \leftharpoonup S_A(m^{(2)2}a_2) \\ &= m^{(1)} \leftharpoonup (m^{(2)1} \rightharpoonup a_1)S_A(m^{(2)2} \rightharpoonup a_2)S_A(m^{(2)3}) \\ &= m^{(1)} \leftharpoonup (m^{(2)1} \rightharpoonup (a_1 S_A(a_2)))S_A(m^{(2)2}) \\ &= \epsilon_A(a)m^{(1)} \leftharpoonup S_A(m^{(2)}) \end{aligned}$$

We now prove that f and g are mutually inverse. It is obvious that $f \circ g = id$. On the other hand, we have by the preceding calculation:

$$\begin{aligned} (g \circ f)(v \otimes a) = p((v \leftharpoonup a)^{(1)}) \otimes (v \leftharpoonup a)^{(2)} &= p(v^{(1)} \leftharpoonup (v^{(2)1} \rightharpoonup a_1)) \otimes v^{(2)2}a_2 \\ &= p(v^{(1)})\epsilon_A(v^{(2)1} \rightharpoonup a_1) \otimes v^{(2)2}a_2 \\ &= p(v^{(1)}) \otimes v^{(2)}a = p(v) \otimes a \end{aligned}$$

It is obvious that f is A-linear, and therefore its inverse g is also A-linear. Similarly, its obvious that g is A-colinear, and therefore f is also A-colinear. $\square$

2.9 We shall now see that the dual vector space of a finite dimensional Yetter-Drinfel'd Hopf algebra is a Yetter-Drinfel'd Hopf module:

Proposition Suppose that A is a finite dimensional Yetter-Drinfel'd Hopf algebra. Choose a basis $a_{(1)}, \ldots, a_{(n)}$ of A and denote the corresponding dual basis by $a^{(1)*}, \ldots, a^{(n)*}$.

1. The dual vector space A^* becomes a left Yetter-Drinfel'd module over H by the cooperation:

$$\delta^H_{A^*}(f) = \sum_{i=1}^n S_H(a_{(i)}{}^1) \otimes f(a_{(i)}{}^2)a^{(i)*}$$

and the operation:

$$(h \rightharpoonup f)(a) = f(S_H^{-1}(h) \rightharpoonup a)$$

for $f \in A^*$, $a \in A$ and $h \in H$.

2. A^* is a right Yetter-Drinfel'd Hopf module with the action:

$$\phi_{A^*} : A^* \otimes A \to A^*, f \otimes a \mapsto (f \leftharpoonup a) = \sum_{i=1}^n a^{(i)*} f^2(a_{(i)}S_A(f^1 \rightharpoonup a))$$

and the coaction:

$$\delta^A_{A^*} : A^* \to A^* \otimes A, f \mapsto f^{(1)} \otimes f^{(2)} = \sum_{i=1}^{n} a^{(i)*} \otimes a_{(i)1} f(a_{(i)2})$$

Proof. That A^* becomes a left Yetter-Drinfel'd module in this way follows from subsections 2.3 and 2.5, because the dual vector space is a right Yetter-Drinfel'd module over H and therefore also a left Yetter-Drinfel'd module over $H^{op\ cop}$. The specified structure now arises from applying the isomorphism $S_H : H^{op\ cop} \to H$. To prove the second statement, we note first that the mappings

$$ev : A \otimes A^* \to K, a \otimes f \mapsto f(a)$$
$$db : K \to A^* \otimes A, \lambda \mapsto \lambda \sum_{i=1}^{n} a^{(i)*} \otimes a_{(i)}$$

are linear and colinear over H, as one can verify by direct calculation. This implies that ϕ_{A^*} and $\delta^A_{A^*}$ are also linear and colinear over H, since they can be written as the composition of H-linear and colinear maps:

$$\phi_{A^*} = (id_{A^*} \otimes ev) \circ (id_{A^*} \otimes \mu_A \otimes id_{A^*}) \circ (db \otimes S_A \otimes id_{A^*}) \circ \sigma_{A^*,A}$$
$$\delta^A_{A^*} = (id_{A^*} \otimes id_A \otimes ev) \circ (id_{A^*} \otimes \Delta_A \otimes id_A) \circ (db \otimes id_{A^*})$$

Now A^* is a comodule over A because we have:

$$\begin{aligned}
\delta^A_{A^*}(f^{(1)}) \otimes f^{(2)} &= \sum_{i,j=1}^{n} a^{(i)*} \otimes a_{(i)1} a^{(j)*}(a_{(i)2}) \otimes a_{(j)1} f(a_{(j)2}) \\
&= \sum_{i=1}^{n} a^{(i)*} \otimes a_{(i)1} \otimes a_{(i)2} f(a_{(i)3}) \\
&= f^{(1)} \otimes \Delta_A(f^{(2)})
\end{aligned}$$

To prove that A^* is a right A-module, we calculate:

$$\begin{aligned}
(f \leftharpoonup a) \leftharpoonup a' &= \sum_{i=1}^{n} a^{(i)*} (f \leftharpoonup a)^2 (a_{(i)} S_A((f \leftharpoonup a)^1 \rightharpoonup a')) \\
&= \sum_{i=1}^{n} a^{(i)*} (f^2 \leftharpoonup a^2)(a_{(i)} S_A(f^1 a^1 \rightharpoonup a')) \\
&= \sum_{i,j=1}^{n} a^{(i)*} a^{(j)*} (a_{(i)} S_A(f^1 a^1 \rightharpoonup a')) f^3 (a_{(j)} S_A(f^2 \rightharpoonup a^2)) \\
&= \sum_{i=1}^{n} a^{(i)*} f^3 (a_{(i)} S_A(f^1 a^1 \rightharpoonup a') S_A(f^2 \rightharpoonup a^2)) \\
&= \sum_{i=1}^{n} a^{(i)*} f^2 (a_{(i)} [f^1 \rightharpoonup (S_A(a^1 \rightharpoonup a') S_A(a^2))])
\end{aligned}$$

$$= \sum_{i=1}^{n} a^{(i)*} f^2(a_{(i)}(f^1 \rightarrow S_A(aa'))) = f \leftarrow (aa')$$

Finally, we have to verify the Yetter-Drinfel'd Hopf module condition. Note first that this condition, if written without Sweedler notation, reads as follows:

$$\delta^A_{A^*} \circ \phi_{A^*} = (\phi_{A^*} \otimes \mu_A) \circ (id_{A^*} \otimes \sigma_{A,A} \otimes id_A) \circ (\delta^A_{A^*} \otimes \Delta_A)$$

where μ_A denotes the multiplication of A. Since we have for $a, b \in A$ and $f \in A^*$, using the Yetter-Drinfel'd condition and the colinearity of ev, that:

$$\begin{aligned}
&(aS_A(f^1 \rightarrow b_1))_1 b_2 f^2((aS_A(f^1 \rightarrow b_1))_2) \\
&\quad = a_1({a_2}^1 \rightarrow (S_A(f^1 \rightarrow b_1))_1) b_2\ f^2({a_2}^2 S_A(f^1 \rightarrow b_1)_2) \\
&\quad = a_1({a_2}^1 f^1 \rightarrow S_A({b_1}^1 \rightarrow b_2)) b_3\ f^3({a_2}^2 (f^2 \rightarrow S_A({b_1}^2))) \\
&\quad = a_1 S_A({a_2}^1 f^1 {b_1}^1 \rightarrow b_2) b_3\ ev({a_2}^2 S_A(f^2 \rightarrow {b_1}^2) \otimes f^3) \\
&\quad = a_1 S_A({a_2}^1 (f^1 \rightarrow b_1)^1 f^2 \rightarrow b_2) b_3\ ev({a_2}^2 S_A((f^1 \rightarrow b_1)^2) \otimes f^3) \\
&\quad = a_1 S_A(b_2) b_3\ ev(a_2 S_A(f^1 \rightarrow b_1) \otimes f^2) \\
&\quad = a_1\ f^2(a_2 S_A(f^1 \rightarrow b))
\end{aligned}$$

we can expand the right hand side as follows:

$$\begin{aligned}
&(\phi_{A^*} \otimes \mu_A) \circ (id_{A^*} \otimes \sigma_{A,A} \otimes id_A) \circ (\delta^A_{A^*} \otimes \Delta_A)(f \otimes a) \\
&\quad = \phi_{A^*}(f^{(1)} \otimes (f^{(2)1} \rightarrow a_1)) \otimes f^{(2)2} a_2 \\
&\quad = \sum_{i=1}^{n} a^{(i)*} f^{(1)2}(a_{(i)} S_A(f^{(1)1} f^{(2)1} \rightarrow a_1)) \otimes f^{(2)2} a_2 \\
&\quad = \sum_{i=1}^{n} a^{(i)*} f^{2(1)}(a_{(i)} S_A(f^1 \rightarrow a_1)) \otimes f^{2(2)} a_2 \\
&\quad = \sum_{i,j=1}^{n} a^{(i)*} a^{(j)*}(a_i S_A(f^1 \rightarrow a_1)) \otimes a_{(j)1} f^2(a_{(j)2}) a_2 \\
&\quad = \sum_{i=1}^{n} a^{(i)*} \otimes (a_{(i)} S_A(f^1 \rightarrow a_1))_1 a_2 f^2((a_{(i)} S_A(f^1 \rightarrow a_1))_2) \\
&\quad = \sum_{i=1}^{n} a^{(i)*} \otimes a_{(i)1}\ f^2(a_{(i)2} S_A(f^1 \rightarrow a))
\end{aligned}$$

On the other hand, we have:

$$\begin{aligned}
\delta^A_{A^*}(f \leftarrow a) &= \sum_{i=1}^{n} a^{(i)*} \otimes a_{(i)1} (f \leftarrow a)(a_{(i)2}) \\
&= \sum_{i,j=1}^{n} a^{(i)*} \otimes a_{(i)1} a^{(j)*}(a_{(i)2})\ f^2(a_{(j)} S_A(f^1 \rightarrow a))
\end{aligned}$$

$$= \sum_{i=1}^{n} a^{(i)*} \otimes a_{(i)1}\ f^2(a_{(i)2} S_A(f^1 \rightharpoonup a))$$

and therefore both sides are equal. □

2.10 We now reach our first goal: To prove the existence and uniqueness of integrals in finite dimensional Yetter-Drinfel'd Hopf algebras. Note that the following Proposition is not new, it is (a variant of) [5], Corollary 5.8 (cf. also [23], section 2, Prop. 3 and 4).

Proposition Suppose that A is a finite-dimensional left Yetter-Drinfel'd Hopf algebra over H. Then we have:

1. There is a nonzero left integral Λ_A of A, a nonzero right integral Γ_A of A, a nonzero left integral λ_A of A^* and a nonzero right integral ρ_A of A^*. Such elements satisfy:

$$\begin{array}{ll} \lambda_A(\Lambda_A) \neq 0 & \lambda_A(\Gamma_A) \neq 0 \\ \rho_A(\Lambda_A) \neq 0 & \rho_A(\Gamma_A) \neq 0 \end{array}$$

2. There is a character $\iota_A : H \to K$, called the integral character of A, and a grouplike element $g_A \in H$, called the integral group element of A, such that we have:

$$\begin{array}{rl} h \rightharpoonup \Lambda_A = \iota_A(h)\Lambda_A & \delta(\Lambda_A) = g_A \otimes \Lambda_A \\ h \rightharpoonup \Gamma_A = \iota_A(h)\Gamma_A & \delta(\Gamma_A) = g_A \otimes \Gamma_A \\ \lambda_A \leftharpoonup h = \iota_A(h)\lambda_A & a^1\lambda_A(a^2) = \lambda_A(a)g_A \\ \rho_A \leftharpoonup h = \iota_A(h)\rho_A & a^1\rho_A(a^2) = \rho_A(a)g_A \end{array}$$

   for every left integral Λ_A and right integral Γ_A of A and for every left integral λ_A and right integral ρ_A of A^*.

3. A is a Frobenius algebra with a nonzero right integral ρ_A of A^* as Frobenius homomorphism. The Casimir element for this Frobenius homomorphism is

$$\Lambda_{A1} \otimes \iota_A^{-1}({\Lambda_{A2}}^1) S_A({\Lambda_{A2}}^2)$$

   where Λ_A is a left integral of A satisfying $\rho_A(\Lambda_A) = 1$ and $\iota_A^{-1} := \iota_A \circ S_H$.

4. The antipode of A is bijective.

Proof. This will be proved in several steps:

(1) Using the right Yetter-Drinfel'd Hopf module structure from Proposition 2.9, we see that the coinvariants are nothing but the left integrals, since we have for $f \in A^*$:

$$\begin{aligned}\delta^A_{A^*}(f) = f \otimes 1 \;\Leftrightarrow\; &\forall a \in A:\; f^{(1)}(a) f^{(2)} = f(a)1 \\ \Leftrightarrow\; &\forall a \in A:\; \sum_{i=1}^n a^{(i)*}(a) a_{(i)1} f(a_{(i)2}) = f(a)1 \\ \Leftrightarrow\; &\forall a \in A:\; a_1 f(a_2) = f(a)1\end{aligned}$$

where $a_{(1)}, \ldots, a_{(n)}$ is a basis of A with dual basis $a^{(1)*}, \ldots, a^{(n)*}$. Since the mapping

$$A^{*coA} \otimes A \rightarrow A^*, f \otimes a \mapsto (f \leftharpoonup a)$$

is a bijection by Proposition 2.8, the dimensions of both sides must be equal: $dim(A^{*coA} \otimes A) = dim A^*$, and therefore $dim A^{*coA} = 1$.

(2) If $\lambda_A \in A^*$ is a left integral, then for all $h \in H$ we have that $\lambda_A \leftarrow h$ is again a left integral:

$$\begin{aligned}a_1(\lambda_A \leftarrow h)(a_2) &= S_H(h_1)h_2 \rightarrow a_1 \lambda_A(h_3 \rightarrow a_2) \\ &= S_H(h_1) \rightarrow 1\; \lambda_A(h_2 \rightarrow a) = 1\; \lambda_A(h \rightarrow a)\end{aligned}$$

Since the space of integrals is one dimensional, we have $\lambda_A \leftarrow h = \iota_A(h)\lambda_A$ for some number $\iota_A(h)$. It is obvious that ι_A is an algebra homomorphism.

Now write:

$$\delta^H_{A^*}(\lambda_A) = \sum_{j=1}^l h_{(j)} \otimes f_{(j)} \in H \otimes A^*,$$

where $h_{(1)}, \ldots, h_{(l)}$ as well as $f_{(1)}, \ldots, f_{(l)}$ are linearly independent. We then have:

$$\delta^H_{A^*}(\lambda_A) = \sum_{j=1}^l h_{(j)} \otimes f_{(j)} = \sum_{i=1}^n S_H({a_{(i)}}^1) \otimes \lambda_A({a_{(i)}}^2) a^{(i)*}$$

Because we have:

$$\begin{aligned}S_H({a_2}^1) \otimes a_1 \lambda_A({a_2}^2) &= S_H({a_2}^1) S_H({a_1}^1) {a_1}^2 \otimes {a_1}^3 \lambda_A({a_2}^2) \\ &= S_H({a_1}^1 {a_2}^1) {a_1}^2 \otimes {a_1}^3 \lambda_A({a_2}^2) \\ &= S_H(a^1) {{a^2}_1}^1 \otimes {{a^2}_1}^2 \lambda_A({a^2}_2) \\ &= S_H(a^1) \otimes 1\; \lambda_A(a^2)\end{aligned}$$

this implies:

$$\begin{aligned}\sum_{j=1}^l h_{(j)} \otimes a_1 f_{(j)}(a_2) &= \sum_{i=1}^n S_H({a_{(i)}}^1) \otimes a_1 \lambda_A({a_{(i)}}^2) a^{(i)*}(a_2) \\ &= S_H({a_2}^1) \otimes a_1 \lambda_A({a_2}^2)\end{aligned}$$

$$= S_H(a^1) \otimes 1\ \lambda_A(a^2) = \sum_{i=1}^n S_H({a_{(i)}}^1) \otimes 1\ \lambda_A({a_{(i)}}^2) a^{(i)*}(a)$$
$$= \sum_{j=1}^l h_{(j)} \otimes 1\ f_{(j)}(a)$$

Since $h_{(1)}, \dots, h_{(l)}$ are linearly independent, we can compare coefficients and conclude that $a_1 f_{(j)}(a_2) = 1 f_{(j)}(a)$. This proves that $f_{(j)}$ is itself a left integral, and therefore must be a multiple a of λ_A, that is $f_{(j)} = \mu_j \lambda_A$ for some $\mu_j \in K$. If we define: $\tilde{g}_A = \sum_{j=1}^l \mu_j h_{(j)}$, we see that $\delta^H_{A^*}(\lambda_A) = \tilde{g}_A \otimes \lambda_A$. Therefore we have:

$$\tilde{g}_A \lambda_A(a) = \sum_{i=1}^n S_H({a_{(i)}}^1) \lambda_A({a_{(i)}}^2) a^{(i)*}(a) = S_H(a^1)\lambda_A(a^2)$$

The comodule axiom now implies that $\tilde{g}_A$ is a grouplike element. Now define: $g_A := \tilde{g}_A^{-1}$.

(3) We know from the structure theorem for Yetter-Drinfel'd Hopf modules that for a nonzero left integral λ_A, the map

$$A \to A^*, a \mapsto (\lambda_A \leftharpoonup a)$$

is bijective. Now we have:

$$(\lambda_A \leftharpoonup a)(a') = \sum_{i=1}^n a^{(i)*}(a') \lambda_A^2(a_{(i)} S_A(\lambda_A^1 \rightharpoonup a))$$
$$= \lambda_A^2(a' S_A(\lambda_A^1 \rightharpoonup a)) = \lambda_A(a' S_A(g_A^{-1} \rightharpoonup a))$$

Therefore, if $S_A(a) = 0$, then we have $\lambda_A \leftharpoonup a = 0$, and therefore $a = 0$.

(4) The fact that the map $A \to A^*, a \mapsto (\lambda_A \leftharpoonup a)$ is bijective implies that the bilinear form

$$A \times A \to A, (a, a') \mapsto (\lambda_A \leftharpoonup a)(a') = \lambda_A(a' S_A(g_A^{-1} \rightharpoonup a))$$

is nondegenerate. This implies that also the bilinear form

$$A \times A \to A, (a, a') \mapsto \lambda_A(aa')$$

is nondegenerate, that is, A is a Frobenius algebra with respect to the Frobenius homomorphism λ_A. We therefore conclude from Proposition 2.7 that there is a nonzero left integral Λ_A and a nonzero right integral Γ_A that are unique up to scalar multiples. We know the proof of that Proposition that we have:

$$\lambda_A(\Lambda_A) \neq 0 \qquad \lambda_A(\Gamma_A) \neq 0$$

Since we know from subsection 2.3 that A^* is right Yetter-Drinfel'd Hopf algebra, and therefore from subsection 2.5 that $A^{*op\ cop}$ is a left Yetter-Drinfel'd Hopf algebra over $H^{op\ cop}$, we conclude that $A^{*op\ cop}$ contains a nonzero left integral ρ_A which obviously is a right integral in A^* that also does not vanish on the integrals of A. Therefore, the first assertion in the Proposition is now proved.

(5) We now proceed to prove the second assertion. We have:

$$\begin{aligned}\lambda_A(a(h \rightarrow \Lambda_A)) &= \lambda_A((h_2 S_H^{-1}(h_1) \rightarrow a)(h_3 \rightarrow \Lambda_A)) \\ &= (\lambda_A \leftarrow h_2)((S_H^{-1}(h_1) \rightarrow a)\Lambda_A) \\ &= \iota_A(h_2)\epsilon_A(S_H^{-1}(h_1) \rightarrow a)\lambda_A(\Lambda_A) \\ &= \iota_A(h)\lambda_A(a\Lambda_A)\end{aligned}$$

This implies $h \rightarrow \Lambda_A = \iota_A(h)\Lambda_A$, since the bilinear form considered above was nondegenerate. By a similar calculation, we can show that $h \rightarrow \Gamma_A = \iota_A(h)\Gamma_A$ and in turn that $\rho_A \leftarrow h = \iota_A(h)\rho_A$.

We have already seen the equality $a^1\lambda_A(a^2) = g_A\lambda_A(a)$. This implies:

$$\begin{aligned}\Lambda_A^1\lambda_A(a\Lambda_A^2) &= S_H(a^1)a^2\Lambda_A^1\lambda_A(a^3\Lambda_A^2) = S_H(a^1)g_A\lambda_A(a^2\Lambda_A) \\ &= \epsilon_A(a)g_A\lambda_A(\Lambda_A) = g_A\lambda_A(a\Lambda_A)\end{aligned}$$

This implies that $\delta(\Lambda_A) = g_A \otimes \Lambda_A$. Similarly, we can prove that $\delta(\Gamma_A) = g_A \otimes \Gamma_A$. The fact that $a^1\rho_A(a^2) = g_A\rho_A(a)$ can be deduced by applying the preceding arguments to $A^{*\,op\,cop}$, which is a left Yetter-Drinfel'd module over $H^{op\,cop}$ by Lemma 2.3 and Lemma 2.5.

(6) We shall now prove part (3) of the Proposition. Applying the comultiplication to the equation $a\Lambda_A = \epsilon_A(a)\Lambda_A$ yields:

$$a_1({a_2}^1 \rightarrow \Lambda_{A1}) \otimes {a_2}^2\Lambda_{A2} = \epsilon_A(a)\Lambda_{A1} \otimes \Lambda_{A2}.$$

We therefore have:

$$\begin{aligned}\epsilon_A(a)\Lambda_{A1} \otimes S_A(\Lambda_{A2}) &= a_1({a_2}^1 \rightarrow \Lambda_{A1}) \otimes S_A({a_2}^2\Lambda_{A2}) \\ &= a_1({a_2}^1 \rightarrow \Lambda_{A1}) \otimes S_A({a_2}^2 \rightarrow \Lambda_{A2})S_A({a_2}^3) \\ &= a_1\Lambda_{A1} \otimes S_A(\Lambda_{A2})S_A(\iota_A({a_2}^1){a_2}^2)\end{aligned}$$

Convolution inversion yields:

$$S_A(a)\Lambda_{A1} \otimes S_A(\Lambda_{A2}) = \Lambda_{A1} \otimes S_A(\Lambda_{A2})S_A(\iota_A(a^1)a^2)$$

If we apply the mapping $a \mapsto \iota_A^{-1}(a^1)a^2$ to the second tensorand in this equation, we get:

$$S_A(a)\Lambda_{A1} \otimes \iota_A^{-1}({\Lambda_{A2}}^1)S_A({\Lambda_{A2}}^2) = \Lambda_{A1} \otimes \iota_A^{-1}({\Lambda_{A2}}^1)S_A({\Lambda_{A2}}^2)S_A(a)$$

Since the antipode of A is bijective, we can replace $S_A(a)$ by a in this equation. We now have:

$$\rho_A(a\Lambda_{A1})\iota_A^{-1}({\Lambda_{A2}}^1)S_A({\Lambda_{A2}}^2) = \rho_A(\Lambda_{A1})\iota_A^{-1}({\Lambda_{A2}}^1)S_A({\Lambda_{A2}}^2)a = a$$

This proves the assertion. $\square$

2.11 The integral character and the integral group element are not totally unrelated: Conjugation by the integral group element is the same as coconjugation by the integral character. Neither, they can be arbitrary characters or grouplike elements, they have to be central among the other grouplikes. Recall from [19], Example 9.1.4 that the characters are precisely the grouplike elements in the finite dual H° of H.

Theorem Suppose that ι_A and g_A are the integral character resp. the integral group element of a finite dimensional left Yetter-Drinfel'd Hopf algebra A. We have:

1. $\forall\, h \in H : g_A h g_A^{-1} = \iota_A^{-1}(h_1) h_2 \iota_A(h_3)$
2. $g_A \in Z(G(H))$, the center of $G(H)$.
3. $\iota_A \in Z(G(H^\circ))$

Proof. Select a right integral $\rho_A \in A^*$ and a left integral $\Lambda_A \in A$ that satisfy $\rho_A(\Lambda_A) = 1$. The Yetter-Drinfel'd condition implies:

$$(h_1 \rightarrow \Lambda_A)^1 h_2 \rho_A((h_1 \rightarrow \Lambda_A)^2) = h_1 {\Lambda_A}^1 \rho_A(h_2 \rightarrow {\Lambda_A}^2)$$

Now the left hand side is $\iota_A(h_1){\Lambda_A}^1 h_2 \rho_A({\Lambda_A}^2) = \iota_A(h_1) g_A h_2$, whereas the right hand side is $h_1 g_A \rho_A(h_2 \rightarrow \Lambda_A) = h_1 g_A \iota_A(h_2)$. This proves the first assertion. To prove the second assertion, assume that $g \in G(H)$ is a grouplike element. Then we have:

$$g_A g g_A^{-1} = \iota_A^{-1}(g) g \iota_A(g) = g$$

This proves the second statement. The third statement follows from similar considerations. □

2.12 The fact that the space of integrals is one-dimensional leads to the existence of certain elements which we study now. Suppose that A is a finite-dimensional left Yetter-Drinfel'd Hopf algebra over H and that Λ_A resp. Γ_A are nonzero left resp. right integrals of A. Since the spaces of left resp. right integrals are one-dimensional, and since $\Lambda_A a$ resp. $a\Gamma_A$ is obviously again a left resp. right integral, we have:

$$\Lambda_A a = \alpha_A^L(a)\Lambda_A \qquad a\Gamma_A = \alpha_A^R(a)\Gamma_A$$

for some numbers $\alpha_A^L(a)$ and $\alpha_A^R(a)$. α_A^L and α_A^R are obviously characters of A which are independent of the choice of the integrals. They are called the left resp. the right modular function of A. Dually, if λ_A resp. ρ_A are nonzero left resp. right integrals of A^*, we have for all $a \in A$:

$$\lambda_A(a_1)a_2 = \lambda_A(a)a_A^L \qquad a_1\rho_A(a_2) = \rho_A(a)a_A^R$$

for two grouplike elements a_A^L and a_A^R of A. These elements are called the left resp. right modular elements of A. Obviously, the modular functions of A are the modular elements of A^*, whereas the modular elements of A represent the modular functions of A^* in the bidual space.

Proposition Suppose that A is a finite-dimensional left Yetter-Drinfel'd Hopf algebra over H.

1. The antipode of A maps left integrals to right integrals and right integrals to left integrals.
2. α_A^L and α_A^R are H-linear and colinear.
3. We have:

$$\begin{aligned} h \rightharpoonup a_A^L &= \epsilon_H(h) a_A^L \qquad & \delta(a_A^L) &= 1 \otimes a_A^L \\ h \rightharpoonup a_A^R &= \epsilon_H(h) a_A^R \qquad & \delta(a_A^R) &= 1 \otimes a_A^R \end{aligned}$$

   that is, the mapping $K \rightarrow A, \xi \mapsto \xi a_A^L$ is H-linear and colinear (and similarly for a_A^R).
4. $a_A^L = (a_A^R)^{-1}$ and $\alpha_A^L = (\alpha_A^R)^{-1}$, where $(\alpha_A^R)^{-1} = \alpha_A^R \circ S_A$.

Proof. To prove the first assertion, observe that for a left integral Λ_A of A we have:

$$\begin{aligned} \epsilon_A(a) S_A(\Lambda_A) &= S_A(a\Lambda_A) \\ &= S_A(a^1 \rightharpoonup \Lambda_A) S_A(a^2) \\ &= S_A(\Lambda_A) S_A(\iota_A(a^1) a^2) \end{aligned}$$

where ι_A is the integral character from Proposition 2.10. Inserting $\iota_A^{-1}(a^1)a^2$ instead of a and using the colinearity of ϵ_A, we see that $S_A(\Lambda_A)$ is a right integral. A similar calculation shows that the antipode maps right integrals to left integrals.

For the second assertion, we prove that α_A^R is H-linear:

$$\begin{aligned} \alpha_A^R(h_1 \rightharpoonup a)\iota_A(h_2)\Gamma_A &= (h_1 \rightharpoonup a)(h_2 \rightharpoonup \Gamma_A) \\ &= h \rightharpoonup (a\Gamma_A) \\ &= \alpha_A^R(a) h \rightharpoonup \Gamma_A \\ &= \alpha_A^R(a)\iota_A(h)\Gamma_A \end{aligned}$$

and therefore $\alpha_A^R(h_1 \rightharpoonup a)\iota_A(h_2) = \alpha_A^R(a)\iota_A(h)$. Convolution multiplication with ι_A^{-1} on both sides yields the H-linearity of α_A^R. One can show similarly that α_A^R is also colinear, and the same is true for α_A^L. The third assertion is the dualization of the second assertion to $A^{*\,op\,cop}$ over $H^{op\,cop}$ using Lemma 2.3 and Lemma 2.5.

It remains to show the fourth assertion in the Proposition. Select a nonzero left integral Λ_A in A. Using the grouplike element g_A from Proposition 2.10, we calculate:

$$\begin{aligned}\alpha_A^L(a)S_A(\Lambda_A) &= S_A(\Lambda_A a)\\ &= S_A({\Lambda_A}^1 \rightharpoonup a)S_A({\Lambda_A}^2)\\ &= S_A(g_A \rightharpoonup a)S_A(\Lambda_A)\\ &= \alpha_A^R(S_A(g_A \rightharpoonup a))S_A(\Lambda_A)\\ &= \alpha_A^R(S_A(a))S_A(\Lambda_A)\end{aligned}$$

The assertion about the modular elements is, as above, the dualization of the statement about the modular functions. □

2.13 If H is finite-dimensional, there are of course also the modular functions α_H^L and α_A^R as well as the modular elements a_H^L and a_H^R for H. The following simple identity relates them to the integral character and the integral group element:

Proposition We have: $\alpha_H^R(g_A)\iota_A(a_H^R) = 1$

Proof. Select right integrals $\Gamma_H \in H$ and $\rho_H \in H^*$ that satisfy $\rho_H(\Gamma_H) = 1$. We know from Theorem 2.11 that:

$$g_A \iota_A(\Gamma_{H1})\Gamma_{H2} = \Gamma_{H1}\iota_A(\Gamma_{H2})g_A$$

If we apply ρ_H to this equation, the left hand side simplifies to:

$$\begin{aligned}\iota_A(\Gamma_{H1})\rho_H(g_A\Gamma_{H2}) &= \iota_A(g_A^{-1})\iota_A(g_A\Gamma_{H1})\rho_H(g_A\Gamma_{H2})\\ &= \iota_A(g_A^{-1})\alpha_H^R(g_A)\iota_A(\Gamma_{H1})\rho_H(\Gamma_{H2})\\ &= \iota_A(g_A^{-1})\alpha_H^R(g_A)\iota_A(a_H^R)\end{aligned}$$

whereas the right hand side yields:

$$\begin{aligned}\rho_H(\Gamma_{H1}g_A)\iota_A(\Gamma_{H2}) &= \iota_A(g_A^{-1})\rho_H(\Gamma_{H1}g_A)\iota_A(\Gamma_{H2}g_A)\\ &= \iota_A(g_A^{-1})\rho_H(\Gamma_{H1})\iota_A(\Gamma_{H2})\\ &= \iota_A(g_A^{-1})\end{aligned}$$

Multiplication by $\iota_A(g_A)$ yields the assertion. □

2.14 As for ordinary Hopf algebras, there is a version of Maschke's theorem for Yetter-Drinfel'd Hopf algebras. This was already observed by D. Fischman, S. Montgomery and H. J. Schneider (cf. [5], Cor. 5.8), in fact, they prove a more general theorem that holds for the subalgebra of coinvariants that arises from a cleft Hopf algebra surjection. Their proof rests on a nice Lemma of M. Koppinen (cf. [11], Prop. 5.2, p. 442). We also note that it is possible to generalize this theorem to Hopf algebras in fairly general categories, however, it seems that a proof of this result has not yet been published. For the sake of completeness, we include a (very slightly more general) variant of their result here, and give a simple proof that stays completely within the framework of the methods developed in this section.

Proposition Suppose that A is a finite-dimensional left Yetter-Drinfel'd Hopf algebra. Pick a nonzero left integral $\Lambda_A \in A$. Then the following statements are equivalent:

1. A is a semisimple algebra.
2. $\epsilon_A(\Lambda_A) \neq 0$.

In this case, we have $\iota_A = \epsilon_H$, $g_A = 1_H$ and $\alpha_A^L = \alpha_A^R = \epsilon_A$.

Proof. If A is semisimple, then the A-linear map $\epsilon_A : A \rightarrow K$ from the left regular A-module A to the trivial A-module K has a right inverse $K \rightarrow A$. This right inverse maps the unit element of K to a left integral Λ_A of A that obviously satisfies $\epsilon_A(\Lambda_A) \neq 0$. For the converse, suppose that we have a left integral Λ_A that satisfies $\epsilon_A(\Lambda_A) = 1$. Then we have:

$$\iota_A(h)\epsilon_A(\Lambda_A) = \epsilon_A(h \rightharpoonup \Lambda_A) = \epsilon_H(h)\epsilon_A(\Lambda_A)$$
$$g_A\epsilon_A(\Lambda_A) = \Lambda_A^1\epsilon_A(\Lambda_A^2) = 1\epsilon_A(\Lambda_A)$$

In addition, we have $\alpha_A^L(a)\epsilon_A(\Lambda_A)\Lambda_A = \Lambda_A a \Lambda_A = \epsilon_A(a)\epsilon_A(\Lambda_A)\Lambda_A$. In particular, since $\iota_A = \epsilon_H$, we have by Proposition 2.10 that $\Lambda_{A1} \otimes S_A(\Lambda_{A2})$ is a Casimir element. Therefore, if M is an A-module with submodule N, we can build from a K-linear projection $\pi : M \rightarrow N$ a map

$$\tilde{\pi} : M \rightarrow N, m \mapsto \Lambda_{A1}\pi(S_A(\Lambda_{A2})m)$$

which is obviously still a projection but in addition is A-linear:

$$\tilde{\pi}(am) = \Lambda_{A1}\pi(S_A(\Lambda_{A2})am) = a\Lambda_{A1}\pi(S_A(\Lambda_{A2})m) = a\tilde{\pi}(m)$$

Therefore, A is semisimple. $\Box$

Using Lemma 2.3 and Lemma 2.5, we can easily derive a dualization:

Corollary Suppose that A is a finite-dimensional left Yetter-Drinfel'd Hopf algebra. Pick a nonzero right integral $\rho_A \in A^*$. Then the following statements are equivalent:

1. A is a cosemisimple coalgebra.
2. $\rho_A(1_A) \neq 0$.

In this case, we have $\iota_A = \epsilon_H$, $g_A = 1_H$ and $a_A^L = a_A^R = 1_A$.

3 Ribbon transformations and monoidal transformations

3.1 In the further analysis of the properties of integrals in Yetter-Drinfel'd Hopf algebras we shall encounter several natural transformations between certain quasisymmetric monoidal functors in the category of Yetter-Drinfel'd modules. Since their properties are better understood from a more general viewpoint, we include here some basic remarks on monoidal categories. Our basic example continues to be the category of left Yetter-Drinfel'd modules considered in the preceding section, and therefore we shall also in this section denote by H a Hopf algebra with bijective antipode.

3.2 Suppose that $\mathcal{C}$ and $\mathcal{D}$ are quasisymmetric monoidal categories and that $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{D}$ are two quasisymmetric monoidal functors (cf. [9],p. 25). Such functors come, by definition, along with certain natural transformations

$$\begin{aligned} \phi_{2,A,B} &: FA \otimes FB \to F(A \otimes B) \qquad & \phi_0 &: E_{\mathcal{D}} \to FE_{\mathcal{C}} \\ \psi_{2,A,B} &: GA \otimes GB \to G(A \otimes B) \qquad & \psi_0 &: E_{\mathcal{D}} \to GE_{\mathcal{C}} \end{aligned}$$

that satisfy their defining conditions (cf. [9]). Here $E_{\mathcal{C}}$ and $E_{\mathcal{D}}$ denote the neutral objects of the monoidal categories $\mathcal{C}$ and $\mathcal{D}$. We consider two kinds of natural transformations $\theta : F \to G$:

Definition Suppose that $\theta : F \to G$ is a natural transformation between quasisymmetric monoidal functors.

1. θ is called a monoidal transformation (cf. [9], p. 38) if the following two diagrams commute:

$$\begin{tikzcd} FA \otimes FB \arrow[r, "\phi_2"] \arrow[d, "\theta_A \otimes \theta_B"'] & F(A \otimes B) \arrow[d, "\theta_{A \otimes B}"] \\ GA \otimes GB \arrow[r, "\psi_2"'] & G(A \otimes B) \end{tikzcd}$$

$$
\begin{tikzcd}
E_{\mathcal{D}} \arrow[r, "\phi_0"] \arrow[d, "1_{E_{\mathcal{D}}}"'] & FE_{\mathcal{C}} \arrow[d, "\theta_{E_{\mathcal{C}}}"] \\
E_{\mathcal{D}} \arrow[r, "\psi_0"'] & GE_{\mathcal{C}}
\end{tikzcd}
$$

2. θ is called a ribbon transformation if the following two diagrams commute:

$$
\begin{tikzcd}
FA \otimes FB \arrow[r, "\phi_2"] \arrow[d, "\theta_A \otimes \theta_B"'] & F(A \otimes B) \arrow[r, "F(\sigma_{B,A} \circ \sigma_{A,B})"] & F(A \otimes B) \arrow[d, "\theta_{A \otimes B}"] \\
GA \otimes GB \arrow[rr, "\psi_2"'] & & G(A \otimes B)
\end{tikzcd}
$$

$$
\begin{tikzcd}
E_{\mathcal{D}} \arrow[r, "\phi_0"] \arrow[d, "1_{E_{\mathcal{D}}}"'] & FE_{\mathcal{C}} \arrow[d, "\theta_{E_{\mathcal{C}}}"] \\
E_{\mathcal{D}} \arrow[r, "\psi_0"'] & GE_{\mathcal{C}}
\end{tikzcd}
$$

This commutativity is of course required for all objects $A, B \in \mathcal{C}$.

The notion of a ribbon transformation is closely related to the notion of a twist (cf. [9], Def. 6.1, p. 65): If F and G are the identity functors, then invertible ribbon transformations are twists in the category $\mathcal{C}'$ where σ is replaced by σ^{-1} (cf. [9], Def. 2.1, p. 33).

3.3 In our context, monoidal functors arise in the following way: Suppose that $\varphi : H \rightarrow H$ is a Hopf algebra automorphism and that V is a left Yetter-Drinfel'd module over H. Then we can use the isomorphism to introduce a new Yetter-Drinfel'd module structure on V in the following way: The new action is given as:

$$h \hookrightarrow v := \varphi(h) \rightarrow v$$

and the new coaction is given as:

$$\delta_\varphi(v) := \varphi^{-1}(v^1) \otimes v^2$$

If V is regarded as a Yetter-Drinfel'd module in this way, we shall denote it by V_φ. What we have established is a functor

$$F_\varphi : {}^H_H YD \rightarrow {}^H_H YD,\ V \mapsto V_\varphi$$

of the category of Yetter-Drinfel'd modules which is the identity on morphisms. It is easy to see that F_φ is strictly monoidal and strictly quasisymmetric, i. e. we have:

$$F_\varphi(V \otimes W) = F_\varphi(V) \otimes F_\varphi(W)$$
$$F_\varphi(\sigma_{V,W}) = \sigma_{F_\varphi(V),F_\varphi(W)}$$

3.4 Now, we look at a special case. Suppose that $g \in H$ is a grouplike element and that $\gamma : H \to K$ is a character. We consider the composition

$$\varphi : H \to H,\ h \mapsto \gamma(h_1) g h_2 g^{-1} \gamma^{-1}(h_3)$$

of the conjugation with g and the coconjugation with γ^{-1}. It is easy to verify that φ is a Hopf algebra isomorphism.

Now suppose that V is a left Yetter-Drinfel'd module over H. We introduce the map:

$$\psi_V : V \to V_\varphi,\ v \mapsto \gamma(v^1) g \rightharpoonup v^2$$

This now turns out to be an example for our abstract notion:

Proposition ψ is a monoidal transformation from the identity functor on the category of Yetter-Drinfel'd modules to the functor F_φ.

Proof. We have to prove that ψ_V is a morphism inside the category, that is, that ψ_V is H-linear and colinear:

$$\psi_V(h \rightharpoonup v) = \varphi(h) \rightharpoonup \psi_V(v) \qquad \delta_V(\psi_V(v)) = \varphi(v^1) \otimes \psi_V(v^2)$$

We only prove the first equality and leave the second one as an easy exercise. We have:

$$\begin{aligned}\psi_V(h \rightharpoonup v) &= \gamma((h \rightharpoonup v)^1) g \rightharpoonup (h \rightharpoonup v)^2 \\ &= \gamma(h_1 v^1 S_H(h_3)) g h_2 \rightharpoonup v^2 \\ &= \gamma(h_1)\gamma(v^1)\gamma^{-1}(h_3) g h_2 g^{-1} g \rightharpoonup v^2 \\ &= \varphi(h) \rightharpoonup \psi_V(v)\end{aligned}$$

It is immediate that ψ is natural, and the assertion that ψ is monoidal means that we have $\psi_{V\otimes W} = \psi_V \otimes \psi_W$, which is also easily verified. $\square$

3.5 We know from [27] that, in general, there is no twist on the category of Yetter-Drinfel'd modules, and therefore no invertible ribbon transformation from the identity functor to itself. However, there is a ribbon transformation between the identity functor and the functor $F_{S_H^2}$. For every left Yetter-Drinfel'd module V over H, we define:

$$\theta_V : V \to V_{S_H^2},\ v \mapsto (S_H(v^1) \rightharpoonup v^2)$$

It is easy to see that θ_V is an isomorphism with inverse:

$$\theta_V^{-1}(v) = S_H^{-2}(v^1) \rightharpoonup v^2$$

Proposition θ is a ribbon transformation from the identity functor to $F_{S_H^2}$.

Proof. As above, we have to prove first that θ_V is H-linear and colinear, that is, that we have:

$$\theta_V(h \rightharpoonup v) = S_H^2(h) \rightharpoonup \theta_V(v) \qquad \delta_V(\theta_V(v)) = S_H^2(v^1) \otimes \theta_V(v^2)$$

We prove the first formula, leaving the second one as an exercise:

$$\begin{aligned}
\theta_V(h \rightharpoonup v) &= S_H((h \rightharpoonup v)^1) \rightharpoonup (h \rightharpoonup v)^2 \\
&= S_H(h_1 v^1 S_H(h_3)) \rightharpoonup (h_2 \rightharpoonup v^2) \\
&= S_H^2(h_3) S_H(v^1) S_H(h_1) h_2 \rightharpoonup v^2 \\
&= S_H^2(h) \rightharpoonup \theta_V(v)
\end{aligned}$$

To prove that θ really is a ribbon transformation, that is, that we have

$$\theta_{V \otimes W} \circ \sigma_{W,V} \circ \sigma_{V,W} = \theta_V \otimes \theta_W$$

is slightly more cumbersome:

$$\begin{aligned}
(\theta_V \otimes \theta_W) \circ \sigma_{V,W}^{-1} \circ \sigma_{W,V}^{-1}(v \otimes w) &= (\theta_V \otimes \theta_W) \circ \sigma_{V,W}^{-1}(w^2 \otimes S_H^{-1}(w^1) \rightharpoonup v) \\
&= (\theta_V \otimes \theta_W)((S_H^{-1}(w^1) \rightharpoonup v)^2 \otimes S_H^{-1}((S_H^{-1}(w^1) \rightharpoonup v)^1) \rightharpoonup w^2) \\
&= (\theta_V \otimes \theta_W)(S_H^{-1}(w^2) \rightharpoonup v^2 \otimes S_H^{-1}(S_H^{-1}(w^3) v^1 w^1) \rightharpoonup w^4) \\
&= (\theta_V \otimes \theta_W)(S_H^{-1}(w^2) \rightharpoonup v^2 \otimes S_H^{-1}(v^1 w^1) S_H^{-2}(w^3) \rightharpoonup w^4) \\
&= (\theta_V \otimes \theta_W)((S_H^{-1}(w^2) \rightharpoonup v^2) \otimes (S_H^{-1}(v^1 w^1) \rightharpoonup \theta_W^{-1}(w^3))) \\
&= (S_H(w^2) \rightharpoonup \theta_V(v^2)) \otimes (S_H(v^1 w^1) \rightharpoonup w^3) \\
&= (S_H(v^2 w^2) \rightharpoonup v^3) \otimes (S_H(v^1 w^1) \rightharpoonup w^3) \\
&= \theta_{V \otimes W}(v \otimes w)
\end{aligned}$$

It is easy to see that θ is really natural. $\square$

3.6 Next, we want to understand how the monoidal transformations and the ribbon transformations interact. Suppose that g and g' are two grouplike elements in H, and that $\gamma : H \to K$ and $\gamma' : H \to K$ are two characters, that is, two grouplike elements of the finite dual H°. Then we can form two monoidal transformations ψ and ψ' that are given on a fixed left Yetter-Drinfel'd module V as:

$$\begin{aligned}
\psi_V &: V \to V, v \mapsto (\gamma(v^1) g \rightharpoonup v^2) \\
\psi'_V &: V \to V, v \mapsto (\gamma'(v^1) g' \rightharpoonup v^2)
\end{aligned}$$

If we consider ψ_V and ψ'_V only as K-linear maps, then we see that:

$$\begin{aligned}\psi'_V \circ \psi_V(v) &= \gamma(v^1)\gamma'((g \rightharpoonup v^2)^1)(g' \rightharpoonup (g \rightharpoonup v^2)^2) \\ &= \gamma(v^1)\gamma'(gv^1g^{-1})(g'g \rightharpoonup v^2) \\ &= (\gamma\gamma')(v^1)(g'g \rightharpoonup v^2)\end{aligned}$$

Therefore, we have constructed a group homomorphism:

Proposition $G(H^\circ)^{op} \times G(H) \to GL(V), (\gamma, g) \mapsto \psi_V$ is a group homomorphism.

In addition, we have to understand the relation of these monoidal transformation with the ribbon transformation as K-linear maps:

Lemma $\psi_V \circ \theta_V = \theta_V \circ \psi_V$

Proof. By the H-linearity and colinearity properties of θ_V from Proposition 3.5, we have:

$$\psi_V(\theta_V(v)) = \gamma(S_H^2(v^1))g \rightharpoonup \theta_V(v^2) = \gamma(v^1)\theta_V(S_H^{-2}(g) \rightharpoonup v^2) = \theta_V(\psi(v)) \quad \square$$

These results will be used throughout in section 4.

3.7 If V is finite-dimensional, the ribbon transformation θ_V admits a simple interpretation in terms of duality. We have already mentioned in subsection 2.9 that we can regard the dual vector space V^* as a left Yetter-Drinfel'd module over H, and we also said there that the mappings

$$\begin{aligned}ev_V &: V \otimes V^* \to K, v \otimes f \mapsto f(v) \\ db_V &: K \to V^* \otimes V, \zeta \mapsto \zeta \sum_{i=1}^n v^{(i)*} \otimes v_{(i)}\end{aligned}$$

are H-linear and colinear, where $v_{(1)}, \dots, v_{(n)}$ is a basis of V with dual basis $v^{(1)*}, \dots, v^{(n)*}$. The canonical isomorphism ξ_V between V and its bidual space becomes H-linear and colinear if considered as a map

$$\xi_V : V_{S_H^{-2}} \to V^{**}$$

Up to this identification, θ_V can be built up from the evaluation ev and the coevaluation db:

Proposition $\xi_V \circ F_{S_H^{-2}}(\theta_V^{-1}) = (id_{V^{**}} \otimes ev_V) \circ (\sigma_{V,V^{**}} \otimes id_{V^*}) \circ (id_V \otimes db_{V^*})$

Proof.

$$
\begin{aligned}
(id_{V^{**}} \otimes ev_V) &\circ (\sigma_{V,V^{**}} \otimes id_{V^*}) \circ (id_V \otimes db_{V^*})(v) \\
&= (id_{V^{**}} \otimes ev_V) \circ (\sigma_{V,V^{**}} \otimes id_{V^*})(\sum_{i=1}^{n} v \otimes \xi_V(v_{(i)}) \otimes v^{(i)*}) \\
&= (id_{V^{**}} \otimes ev_V)(\sum_{i=1}^{n} v^1 \rightarrow \xi_V(v_{(i)}) \otimes v^2 \otimes v^{(i)*}) \\
&= v^1 \rightarrow \xi_V(v^2) = \xi_V(S_H^{-2}(v^1) \rightarrow v^2) \qquad \square
\end{aligned}
$$

3.8 It is a very strange fact that it is possible to calculate the powers of θ_V in a unified form. To derive this formula, we first introduce a strange expression: For $h \in H$, define:

$$P_n(h) := \prod_{k=1}^{n} S_H^{2(n-k)+1}(h_k)$$

where by convention $P_0(h) = \epsilon_H(h)1_H$. In the above formula, the index k in h_k denotes a Sweedler index, that is, the $n-1$-times iterated coproduct is denoted by $\bigotimes_{k=1}^{n} h_k$. Using this, we now can calculate the powers of θ_V for a left Yetter-Drinfel'd module V in the following form:

Proposition $\theta_V^n(v) = P_n(v^1) \rightarrow v^2$

Proof. We proceed by induction on n, the case $n = 0$ being obvious and the case $n = 1$ too because we have $P_1(h) = S_H(h)$. It follows directly from the definition that we have $P_{n+1}(h) = S_H^2(P_n(h_1))S_H(h_2)$. Now, we calculate:

$$
\begin{aligned}
\theta_V^{n+1}(v) &= \theta_V(P_n(v^1) \rightarrow v^2) \\
&= S_H^2(P_n(v^1)) \rightarrow \theta_V(v^2) \\
&= S_H^2(P_n(v^1))S_H(v^2) \rightarrow v^3 \\
&= P_{n+1}(v^1) \rightarrow v^2
\end{aligned}
$$

which establishes the inductive step. □

3.9 If H is finite-dimensional, the natural transformations discussed so far can also be understood from the point of view of the Drinfel'd double construction, and in this way they become very simple. First, observe that for any Hopf algebra H and any grouplike element $g \in H$ the conjugation with g:

$$\varphi : H \rightarrow H, \; h \mapsto ghg^{-1}$$

is obviously a Hopf algebra automorphism, and if we define for every H-module V a new module structure V_φ by pullback via φ as in subsection 3.3, then we obtain a monoidal functor from the category of all H-modules into itself. Now, if we define for all H-modules V the map

$$\psi_V : V \to V_\varphi,\ v \mapsto (g \to v)$$

then this is obviously a monoidal transformation from the identity functor on the category of H-modules to this functor.

We now explain how the monoidal transformations constructed in subsection 3.3 can be understood from this point of view. We shall use the Drinfel'd double construction (cf. [3], §13, [19], §10.3, [24], sec. 3). Let $D = D(H^{cop})^{cop}$ denote the Drinfel'd double of the coopposite Hopf algebra H^{cop} with the coopposite coproduct. D, which is isomorphic to $H^* \otimes H$ as a vector space, is a Hopf algebra with multiplication:

$$(p \otimes h)(p' \otimes h') = (h_3 \to p' \leftarrow S_H(h_1))p \otimes h_2 h'$$

where the arrows denote the coregular actions, comultiplication:

$$\Delta_D(p \otimes h) = (p_1 \otimes h_1) \otimes (p_2 \otimes h_2)$$

unit $1_D = \epsilon_H \otimes 1_H$, counit:

$$\epsilon_D(p \otimes h) = p(1_H)\epsilon_H(h)$$

and antipode:

$$S_D(p \otimes h) = (\epsilon_H \otimes S_H(h)) \otimes (S_{H^*}^{-1}(p) \otimes 1_H)$$

D is also quasitriangular with R-matrix:

$$R = \sum_{i=1}^{n} (h^{(i)*} \otimes 1_H) \otimes (\epsilon_H \otimes h_{(i)})$$

where $h_{(1)}, \dots, h_{(n)}$ is a basis of H with dual basis $h^{(1)*}, \dots, h^{(n)*}$.

Now, D is related to our context as follows: It is easy to see that left Yetter-Drinfel'd modules are the same as D-modules: If V is a left Yetter-Drinfel'd module, then V can be turned into a D-module via the action:

$$(p \otimes h) \to v := (h \to v)^2 p((h \to v)^1)$$

In this way, we obtain a functor from the category of left Yetter-Drinfel'd modules into the category of left D-modules which is easily seen to be strictly monoidal, strictly quasisymmetric and an isomorphism of categories. Suppose now that we are given a grouplike element $g \in H$ and a character $\gamma : H \to K$. The element $g_D := (\epsilon_H \otimes g)(\gamma \otimes 1_H)$ is a grouplike element of D because it is the product of two grouplike elements, and we have $g_D \to v = \gamma(v^1) g \to v^2$.

Therefore, the monoidal transformations considered in subsection 3.3 reduce to the action of grouplike elements considered above.

In a similar fashion, the ribbon transformation θ can be interpreted in terms of D. As for every quasitriangular Hopf algebra, we get from the R-matrix an element:

$$u_D = \sum_{i=1}^{n} S_D(\epsilon_H \otimes h_{(i)})(h^{(i)*} \otimes 1_H) = \sum_{i=1}^{n} (\epsilon_H \otimes S_H(h_{(i)}))(h^{(i)*} \otimes 1_H)$$

We then have:

$$u_D \to v = \sum_{i=1}^{n} (S_H(h_{(i)}) \to v^2) h^{(i)*}(v^1) = S_H(v^1) \to v^2 = \theta_V(v)$$

This means that our ribbon transformation coincides with the action of u_D. The fact that θ is a ribbon transformation now turns out to be equivalent to the well-known formula (cf. [19], Theorem 10.1.13, formula (*), p. 181):

$$u_D \otimes u_D = \Delta_D(u_D)(R^{21} R)$$

To see this, we write $R = \sum_i a_{(i)} \otimes b_{(i)}$. We then have:

$$\begin{aligned}
\theta_{V \otimes W} \circ \sigma_{W,V} \circ \sigma_{V,W}(v \otimes w) &= \theta_{V \otimes W} \circ \sigma_{W,V}(\sum_j b_{(j)} \to w \otimes a_{(j)} \to v) \\
&= u_D \to (\sum_{i,j} b_{(i)} a_{(j)} \to v \otimes a_{(i)} b_{(j)} \to w) \\
&= u_D \to v \otimes u_D \to w \\
&= (\theta_V \otimes \theta_W)(v \otimes w)
\end{aligned}$$

3.10 Monoidal transformations and ribbon transformations may in particular be defined for Yetter-Drinfel'd Hopf algebras. Here, they lead to algebra automorphisms in the case of monoidal transformations, and also in the case of ribbon transformations if composed with the square of the antipode:

Proposition Suppose that $g \in H$ is a grouplike element and that $\gamma : H \to K$ is a character. Define:

$$\varphi : H \to H,\ h \mapsto \gamma(h_1) g h_2 g^{-1} \gamma^{-1}(h_3)$$

Suppose that A is a left Yetter-Drinfel'd Hopf algebra.

1. A_φ is again a Yetter-Drinfel'd Hopf algebra with the same multiplication, comultiplication, unit, counit and antipode.
2. The map $\psi_A : A \to A_\varphi,\ a \mapsto (\gamma(a^1) g \to a^2)$ is a Yetter-Drinfel'd Hopf algebra isomorphism.
3. $S_A^2 \circ \theta_A$ is a Yetter-Drinfel'd Hopf algebra endomorphism of A.

Proof. The first statement is rather obvious. To prove the second, we observe first that ψ_A commutes with the multiplication $\mu_A = \mu_{A_\varphi}$:

$$\psi_A \circ \mu_A = \mu_A \circ \psi_{A\otimes A} = \mu_A \circ (\psi_A \otimes \psi_A)$$

Here, the first equality holds because ψ is natural, and the second because ψ is monoidal. By a similar calculation, it follows that ψ_A commutes with the comultiplication. It is clear that it preserves the unit and the counit. Finally, we have $\psi_A \circ S_A = S_A \circ \psi_A$ since ψ_A is natural. This proves the second statement. For the last one, observe that we have:

$$\theta_A \circ \mu_A = \mu_A \circ \theta_{A\otimes A} = \mu_A \circ (\theta_A \otimes \theta_A) \circ \sigma_{A,A}^{-2}$$

Again, the first equality holds because θ is natural, and the second holds because θ is a ribbon transformation. Now, it is well known that we have: $S_A \circ \mu_A = \mu_A \circ (S_A \otimes S_A) \circ \sigma_{A,A}$, and therefore we have:

$$S_A^2 \circ \theta_A \circ \mu_A = S_A^2 \circ \mu_A \circ (\theta_A \otimes \theta_A) \circ \sigma_{A,A}^{-2} = \mu_A \circ (S_A^2 \otimes S_A^2) \circ (\theta_A \otimes \theta_A)$$

It follows again by similar calculations that $S_A \circ \theta_A$ commutes with the comultiplication and preserves the unit and the counit. It commutes with the antipode since θ is natural. $\Box$

3.11 The morphism θ_A also occurs in the formula for the squared antipode of the Radford biproduct (cf. [23], [19],§10.6). Recall that the Radford biproduct construction, which is the analogue of a semidirect product for Hopf algebras, assigns to every Yetter-Drinfel'd Hopf algebra A an ordinary Hopf algebra $A \otimes H$ with multiplication:

$$(a \otimes h)(a' \otimes h') = a(h_1 \rightharpoonup a') \otimes h_2 h'$$

comultiplication:

$$\Delta(a \otimes h) = (a_1 \otimes {a_2}^1 h_1) \otimes ({a_2}^2 \otimes h_2)$$

unit $1_A \otimes 1_H$, counit $\epsilon_A \otimes \epsilon_H$ and antipode:

$$S_{A\otimes H}(a \otimes h) = (1_A \otimes S_H(a^1 h))(S_A(a^2) \otimes 1_H)$$

We have the following formula for the square of the antipode in $A \otimes H$:

Proposition $S_{A\otimes H}^2(a \otimes h) = S_A^2(\theta_A(a)) \otimes S_H^2(h)$

Proof. We have:

$$\begin{aligned}
S_{A\otimes H}^2(a \otimes h) &= S_{A\otimes H}(S_A(a^2) \otimes 1_H) S_{A\otimes H}(1_A \otimes S_H(a^1 h)) \\
&= (1_A \otimes S_H(a^2))(S_A^2(a^3) \otimes 1_H)(1_A \otimes S_H^2(a^1 h)) \\
&= S_A^2(S_H(a^3) \rightharpoonup a^4) \otimes S_H(a^2) S_H^2(a^1) S_H^2(h) \\
&= S_A^2(\theta_A(a)) \otimes S_H^2(h)
\end{aligned}$$

since the antipode is an algebra antihomomorphism. $\Box$

We have to emphasize that the above formula was noted independently and earlier by N. Andruskiewitsch and H.-J. Schneider (cf. [2], p. 7, eq. (4.5)) where also the algebra homomorphism $S_A^2 \circ \theta_A$ is considered in this manner, that is, by restriction from the Radford biproduct.

3.12 Two monoidal transformations that will play an important role in the next section deserve special names. In fact, the sole purpose of this section was to describe their properties from a more abstract point of view.

Definition Suppose that V is a left Yetter-Drinfel'd module over H.

1. If H is finite-dimensional and α_H^R (resp. a_H^R) is the right modular function (resp. modular element), define the right modular transformation as:

$$M_V^R : V \to V_{S_H^4}, v \mapsto \alpha_H^R(v^1)(a_H^R \to v^2)$$

   Similarly, define the left modular transformation as:

$$M_V^L : V \to V_{S_H^{-4}}, v \mapsto \alpha_H^L(v^1)(a_H^L \to v^2)$$

   where α_H^L (resp. a_H^L) denotes the left modular function (resp. modular element).

2. If A is a finite-dimensional left Yetter-Drinfel'd Hopf algebra over H with integral character ι_A and integral group element g_A, we define the integral transformation as:

$$I_V : V \to V, v \mapsto \iota_A(v^1)(g_A \to v^2)$$

We note that a relation of the use of the term 'modular transformation' used here to the use of the same term in [17] is not intended. Similarly, the term 'integral transform' is not intended to mean something like a 'Fourier transform'.

3.13 As we have already explained in subsection 3.10, the composite $S_A^2 \circ \theta_A$ is a Yetter-Drinfel'd Hopf algebra morphism. The following Proposition reformulates in the context of ribbon transformations a nice formula of N. Andruskiewitsch and H. J. Schneider for its trace:

Proposition Suppose that $\rho_A \in A^*$ and $\Gamma_A \in A$ are right integrals satisfying $\rho_A(\Gamma_A) = 1$. Then we have: $Tr(S_A^2 \circ \theta_A) = \rho_A(1)\epsilon_A(\Gamma_A)$

For a proof, we refer to [2], Theorem 7.3. Andruskiewitsch and Schneider also prove that the ordinary trace of $S_A^2 \circ \theta_A$ coincides with the categorical trace of S_A^2. However, this result depends on their definition of the categorical trace: Their definition agrees with the definition of P. J. Freyd and D. N. Yetter (cf. [6], Def. 1.5, p. 160), but not with the refined definition of V. G. Turaev (cf. [36], [37], sec. I.1.5, p. 21). However, since Turaev's definition involves a twist, it is only defined for ribbon categories, and, as we have already noted above, we know from [27] that the category of Yetter-Drinfel'd modules is, in general, not a ribbon category. But we have already explained that the ribbon transformation θ_V can in a way serve as a substitute for this. Therefore, one reasonable definition of the categorical trace of an H-linear and colinear endomorphism $f : V \to V$ of some Yetter-Drinfel'd module V would be the image of the unit in the composition

$$ev \circ (\theta_V^{-1} \otimes id_{V^*}) \circ \sigma_{V^*,V} \circ (id_{V^*} \otimes f) \circ db$$

from K to K, where ev and db are defined as in subsection 3.7. This is justified in particular because a simple calculation shows that this notion of a categorical trace agrees with the ordinary trace.

4 Antipodes and Nakayama automorphisms

4.1 In this section, we prove an analogue of a formula of D. Radford (cf. [22], Prop. 6, p. 347) for the fourth power of the antipode in the case of Yetter-Drinfel'd Hopf algebras. In contrast to Radford's result on ordinary Hopf algebras, the formula involves in the Yetter-Drinfel'd case the modular and the integral transformations as well as the ribbon transformation θ considered in the previous section. The proof of the formula follows a proof of H.-J. Schneider of Radford's result (cf. [30]). In order to prove this formula, we introduce twisted versions of the Nakayama automorphism and derive explicit formulas for them in terms of the square of the antipode. We also consider the interrelation between the twisted Nakayama automorphisms, the modular and integral transformations and the ribbon transformation θ considered in the last section. Along the way, we calculate the eigenvalue of the squared antipode on an integral.

In the whole section, A denotes a finite-dimensional left Yetter-Drinfel'd Hopf algebra over a Hopf algebra H. We assume that the antipode of H is bijective. We shall use the notation of the preceding sections, in particular for the integral character ι_A, the integral group element g_A, and the modular elements and functions a_A^L, a_A^R, α_A^L and α_A^R of A. $\rho_A \in A^*$ and $\Gamma_A \in A$ denote right integrals that satisfy $\rho_A(\Gamma_A) = 1$. M_V^R, M_V^L, I_V and θ_V denote the modular, integral and the ribbon transformation introduced in section 3.

4.2 We have already calculated in Proposition 2.10 the Casimir element of A that arises from a nonzero right integral. However, we shall need two other forms of this element.

Proposition Suppose that $\rho_A \in A^*$ and $\Gamma_A \in A$ are right integrals satisfying $\rho_A(\Gamma_A) = 1$. Then the Casimir element for the Frobenius homomorphism ρ_A is given by:

$$S_A(g_A^{-1} \rightharpoonup \Gamma_{A1})a_A^R \otimes \Gamma_{A2} = S_A^{-1}(\Gamma_{A2}{}^3) \otimes \iota_A(\Gamma_{A2}{}^2)S_H^{-1}(\Gamma_{A2}{}^1) \rightharpoonup \Gamma_{A1}$$

Proof. The first element is the Casimir element of ρ_A because we have:

$$\begin{aligned}
S_A(g_A^{-1} \rightharpoonup \Gamma_{A1})a_A^R\rho_A(\Gamma_{A2}a) &= S_A(S_H(\Gamma_A{}^1) \rightharpoonup \Gamma_A{}^2{}_1)a_A^R\rho_A(\Gamma_A{}^2{}_2a) \\
&= S_A(S_H(\Gamma_A{}^2) \rightharpoonup \Gamma_A{}^3{}_1)\epsilon_H(\Gamma_A{}^1)a_A^R\rho_A(\Gamma_A{}^3{}_2a) \\
&= S_H(\Gamma_A{}^1) \rightharpoonup (S_A(\Gamma_A{}^2{}_1)a_A^R\rho_A(\Gamma_A{}^2{}_2a)) \\
&= S_H(\Gamma_A{}^1) \rightharpoonup (S_A(\Gamma_A{}^2{}_1)\Gamma_A{}^2{}_2(\Gamma_A{}^2{}_3{}^1 \rightharpoonup a_1))\rho_A(\Gamma_A{}^2{}_3{}^2a_2) \\
&= S_H(\Gamma_A{}^1) \rightharpoonup (\Gamma_A{}^2 \rightharpoonup a_1)\rho_A(\Gamma_A{}^3a_2) \\
&= a_1\rho_A(\Gamma_A a_2) = a
\end{aligned}$$

To show that the second element is the Casimir element of ρ_A, we use the skew-antipode equation:

$$\begin{aligned}
&S_A^{-1}(\Gamma_{A2}{}^3)\iota_A(\Gamma_{A2}{}^2)\rho_A((S_H^{-1}(\Gamma_{A2}{}^1) \rightharpoonup \Gamma_{A1})a) \\
&= S_A^{-1}(\Gamma_{A2}{}^3)(\rho_A \leftharpoonup \Gamma_{A2}{}^2)((S_H^{-1}(\Gamma_{A2}{}^1) \rightharpoonup \Gamma_{A1})a) \\
&= S_A^{-1}(\Gamma_{A2}{}^2)\rho_A(\Gamma_{A1}(\Gamma_{A2}{}^1 \rightharpoonup a)) \\
&= S_A^{-1}(\Gamma_{A2}{}^3)(S_H^{-1}(\Gamma_{A2}{}^2) \rightharpoonup \rho_A(\Gamma_{A1}(\Gamma_{A2}{}^1 \rightharpoonup a))1_A) \\
&= S_A^{-1}(\Gamma_{A3}{}^4)(S_H^{-1}(\Gamma_{A3}{}^3) \rightharpoonup \rho_A(\Gamma_{A1}(\Gamma_{A2}{}^1\Gamma_{A3}{}^1 \rightharpoonup a_1))(\Gamma_{A2}{}^2(\Gamma_{A3}{}^2 \rightharpoonup a_2))) \\
&= \rho_A(\Gamma_{A1}(\Gamma_{A2}{}^1\Gamma_{A3}{}^1 \rightharpoonup a_1))S_A^{-1}(\Gamma_{A3}{}^3)(S_H^{-1}(\Gamma_{A3}{}^2) \rightharpoonup \Gamma_{A2}{}^2)a_2 \\
&= \rho_A(\Gamma_{A1}(\Gamma_{A2}{}^1 \rightharpoonup a_1))S_A^{-1}(\Gamma_{A2}{}^2{}_2{}^2)(S_H^{-1}(\Gamma_{A2}{}^2{}_2{}^1) \rightharpoonup \Gamma_{A2}{}^2{}_1)a_2 \\
&= \rho_A(\Gamma_{A1}(\Gamma_{A2}{}^1 \rightharpoonup a_1))\epsilon_A(\Gamma_{A2}{}^2)a_2 \\
&= \rho_A(\Gamma_A a_1)a_2 = a
\end{aligned}$$

where we have used the skew-antipode equation in the seventh equality. □

Let us draw one obvious conclusion. We have seen in Proposition 2.12 that the antipode maps left integrals to right integrals, and similarly maps right integrals to left integrals. Since the space of right integrals is one-dimensional, every right integral is an eigenvector for the squared antipode. It a natural question to ask for the eigenvalue:

Corollary If Λ_A and Γ_A are a left and a right integral respectively, then they are eigenvectors of S_A^2 corresponding to the eigenvalue $\iota_A(g_A)^2\alpha_A^R(a_A^R)$

Proof. Apply $id_A \otimes \epsilon_A$ to the equality in the above Proposition. □

Note that if we set $H = K$, the base field, then Yetter-Drinfel'd Hopf algebras become ordinary Hopf algebras. In this case, the above Proposition reduces to the formula:

$$S_A(\Gamma_{A1})a_A^R \otimes \Gamma_{A2} = S_A^{-1}(\Gamma_{A2}) \otimes \Gamma_{A1}$$

This version of the formula was proved earlier by D. Radford (cf. [26], Thm. 3, p. 595), while the formula for the eigenvalue of the squared antipode on integrals reduces to still another formula of D. Radford (cf. [22], Cor. 5, p. 345).

4.3 By the very definition of a Frobenius algebra, the form

$$(a, a') \mapsto \rho_A(aa')$$

is nondegenerate. Since for a fixed $a \in A$ the mapping $a' \mapsto \rho_A(aa')$ is a linear form on A, there must be an element $\nu_A(a)$ such that we have:

$$\rho_A(aa') = \rho_A(a'\nu_A(a))$$

for all $a' \in A$. This determines a mapping ν_A from A to A which is easily seen to be an algebra automorphism. It is called the Nakayama automorphism of A.

However, from the viewpoint of braided monoidal categories, the above equation involves a twist of a and a' which was not taken to be the twist map inside the category of Yetter-Drinfel'd modules. It must be expected that the analogous mappings involving the braiding of the category also do play an important role, perhaps even have nicer properties. We therefore introduce the twisted Nakayama automorphisms, which exist by a similar reasoning.

Definition Suppose that $\rho_A \in A^*$ is a nonzero right integral.

1. We define the positively twisted Nakayama automorphism $\nu_+ : A \rightarrow A$ to be the unique map satisfying

$$\forall a, a' \in A : \rho_A(a\nu_+(a')) = \rho_A((a^1 \rightarrow a')a^2)$$

2. We define the negatively twisted Nakayama automorphism $\nu_- : A \rightarrow A$ to be the unique map satisfying

$$\forall a, a' \in A : \rho_A(a'\nu_-(a)) = \rho_A(a^2(S_H^{-1}(a^1) \rightarrow a'))$$

4.4 We collect some of the elementary properties of the twisted Nakayama automorphisms. Although they have - in comparison to the ordinary Nakayama automorphism ν_A - the advantage of being H-linear and colinear, they are not algebra homomorphisms in general, but satisfy a kind of doubly twisted multiplicativity, as the antipode or the ribbon transformation θ.

Proposition

1. We have for all $a, a' \in A$:

$$\rho_A(a'^2(\nu_+(S_H^{-1}(a'^1) \rightarrow a))) = \rho_A(aa') = \rho_A((a^1 \rightarrow a')\nu_-(a^2))$$

2. ν_+ and ν_- are H-linear and colinear.

3. The twisted Nakayama automorphisms satisfy:

$$\nu_+ \circ \mu_A = \mu_A \circ (\nu_+ \otimes \nu_+) \circ \sigma_{A,A}^{-2}$$
$$\nu_- \circ \mu_A = \mu_A \circ (\nu_- \otimes \nu_-) \circ \sigma_{A,A}^{2}$$

where σ denotes the quasisymmetry in the category of Yetter-Drinfel'd modules and μ_A denotes the multiplication map of A.

4. We have for all $a, a' \in A$:

$$\rho_A(\nu_+(a)\nu_-(a')) = \rho_A(aa') = \rho_A(\nu_-(a)\nu_+(a'))$$

Proof. We first prove 1. The first equality says, written in terms of maps without using elements, that we have:

$$\rho_A \circ \mu_A \circ (id_A \otimes \nu_+) \circ \sigma_{A,A}^{-1} = \rho_A \circ \mu_A$$

whereas by definition of ν_+ we have:

$$\rho_A \circ \mu_A \circ (id_A \otimes \nu_+) = \rho_A \circ \mu_A \circ \sigma_{A,A}$$

The second equation is of similar difficulty. We shall show now that ν_+ is H-linear:

$$\begin{aligned}
\rho_A(a(h \rightarrow \nu_+(a'))) &= \rho_A((h_2 S_H^{-1}(h_1) \rightarrow a)(h_3 \rightarrow \nu_+(a'))) \\
&= (\rho_A \leftarrow h_2)((S_H^{-1}(h_1) \rightarrow a)\nu_+(a')) \\
&= \iota_A(h_2)\rho_A((S_H^{-1}(h_1) \rightarrow a)\nu_+(a')) \\
&= (\rho_A \leftarrow h_2)(((S_H^{-1}(h_1) \rightarrow a)^1 \rightarrow a')(S_H^{-1}(h_1) \rightarrow a)^2) \\
&= (\rho_A \leftarrow h_4)((S_H^{-1}(h_3) a^1 h_1 \rightarrow a')(S_H^{-1}(h_2) \rightarrow a^2)) \\
&= \rho_A((a^1 h \rightarrow a')a^2) \\
&= \rho_A(a\nu_+(h \rightarrow a'))
\end{aligned}$$

where we have used a variant of the Yetter-Drinfel'd condition in the fifth equality. We shall show now that ν_- is colinear. We have by the first part:

$$\begin{aligned}
a^1 a'^1 \rho_A((a^2 \rightarrow a'^2)\nu_-(a^3)) &= a^1 a'^1 \rho_A(a^2 a'^2) \\
&= g_A \rho_A(aa') = g_A \rho_A((a^1 \rightarrow a')\nu_-(a^2)) \\
&= (a^1 \rightarrow a')^1 \nu_-(a^2)^1 \rho_A((a^1 \rightarrow a')^2 \nu_-(a^2)^2) \\
&= a^1 a'^1 S_H(a^3)\nu_-(a^4)^1 \rho_A((a^2 \rightarrow a'^2)\nu_-(a^4)^2)
\end{aligned}$$

This implies $1_H \rho_A((a^1 \rightarrow a')\nu_-(a^2)) = S_H(a^2)\nu_-(a^3)^1 \rho_A((a^1 \rightarrow a')\nu_-(a^3)^2)$ which in turn implies $1_H \rho_A(a'\nu_-(a)) = S_H(a^1)\nu_-(a^2)^1 \rho_A(a'\nu_-(a^2)^2)$. Hence we finally see:

$$a^1 \rho_A(a'\nu_-(a^2)) = \nu_-(a)^1 \rho_A(a'\nu_-(a)^2)$$

The proof that ν_+ is colinear and ν_- is linear rests on similar calculations. Next, we prove 3. The first equation can be written in the form:

$$\nu_+ \circ \mu_A \circ \sigma_{A,A} = \mu_A \circ (\nu_+ \otimes \nu_+) \circ \sigma^{-1}_{A,A}$$

Using the form of the definition of ν_+ given during the proof of part 1, we now carry out the following calculation:

$$\begin{aligned}
&\rho_A \circ \mu_A \circ (id_A \otimes \nu_+) \circ (id_A \otimes \mu_A) \circ \sigma^{-1}_{A,A\otimes A} \\
&\quad = \rho_A \circ \mu_A \circ (id_A \otimes \nu_+) \circ \sigma^{-1}_{A,A} \circ (\mu_A \otimes id_A) \\
&\quad = \rho_A \circ \mu_A \circ (\mu_A \otimes id_A) \\
&\quad = \rho_A \circ \mu_A \circ (id_A \otimes \mu_A) \\
&\quad = \rho_A \circ \mu_A \circ (id_A \otimes \nu_+) \circ \sigma^{-1}_{A,A} \circ (id_A \otimes \mu_A) \\
&\quad = \rho_A \circ \mu_A \circ (id_A \otimes \nu_+) \circ (\mu_A \otimes id_A) \circ \sigma^{-1}_{A\otimes A,A} \\
&\quad = \rho_A \circ \mu_A \circ (id_A \otimes \mu_A) \circ (id_A \otimes id_A \otimes \nu_+) \circ \sigma^{-1}_{A\otimes A,A} \\
&\quad = \rho_A \circ \mu_A \circ (id_A \otimes \nu_+) \circ \sigma^{-1}_{A,A} \circ (id_A \otimes \mu_A) \circ (id_A \otimes id_A \otimes \nu_+) \circ \sigma^{-1}_{A\otimes A,A} \\
&\quad = \rho_A \circ \mu_A \circ (\mu_A \otimes \nu_+) \circ \sigma^{-1}_{A\otimes A,A} \circ (id_A \otimes id_A \otimes \nu_+) \circ \sigma^{-1}_{A\otimes A,A} \\
&\quad = \rho_A \circ \mu_A \circ (\mu_A \otimes id_A) \circ (id_A \otimes \nu_+ \otimes \nu_+) \circ \sigma^{-2}_{A\otimes A,A}
\end{aligned}$$

Using the Yang-Baxter equation, this can be reduced to:

$$\rho_A \circ \mu_A \circ (id_A \otimes \nu_+) \circ (id_A \otimes \mu_A) = \rho_A \circ \mu_A \circ (id_A \otimes \mu_A) \circ (id_A \otimes \nu_+ \otimes \nu_+) \circ (id_A \otimes \sigma^{-2}_{A,A})$$

which yields the first equation in 3. by nondegeneracy. The second equation can be shown similarly. The last part follows by inserting the definitions. □

4.5 We now proceed to derive explicit formulas for the twisted Nakayama automorphisms. We shall need the following Lemma, which tells us how the mappings $(\rho_A \circ \mu_A) \otimes id_A$ and $id_A \otimes (\rho_A \circ \mu_A)$ are interrelated:

Lemma We have for all $a, a' \in A$:

1. $\rho_A(a^2 a'_2) a^R_A S^{-1}_A(a^1 \rightarrow a'_1) = a_1 \rho_A(a_2 a')$
2. $\rho_A(a {a'_2}^2) a^R_A S^{-1}_A(S^{-1}_H({a'_2}^1) \rightarrow a'_1) = g^{-1}_A \rightarrow a_1 \rho_A(a_2 a')$

Proof. The first statement follows from the following calculation:

$$
\begin{aligned}
&\rho_A(a^2 a'_2) a_A^R S_A^{-1}(a^1 \rightharpoonup a'_1) \\
&\qquad = \rho_A(a^2 a'_2) a_A^{R2} S_A^{-1}(S_H^{-1}(a_A^{R1}) a^1 \rightharpoonup a'_1) \\
&\qquad = \rho_A({a^2}_2{}^2 a'_3)({a^2}_1({a^2}_2{}^1 \rightharpoonup a'_2))^2 S_A^{-1}(S_H^{-1}(({a^2}_1({a^2}_2{}^1 \rightharpoonup a'_2))^1) a^1 \rightharpoonup a'_1) \\
&\qquad = \rho_A({a_2}^3 a'_3)({a_1}^2({a_2}^2 \rightharpoonup a'_2))^2 S_A^{-1}(S_H^{-1}(({a_1}^2({a_2}^2 \rightharpoonup a'_2))^1) {a_1}^1 {a_2}^1 \rightharpoonup a'_1) \\
&\qquad = \rho_A({a_2}^5 a'_3) {a_1}^3({a_2}^3 \rightharpoonup {a'_2}^2) S_A^{-1}(S_H^{-1}({a_1}^2 {a_2}^2 {a'_2}^1 S_H({a_2}^4)) {a_1}^1 {a_2}^1 \rightharpoonup a'_1) \\
&\qquad = \rho_A({a_2}^3 a'_3) a_1({a_2}^1 \rightharpoonup {a'_2}^2) S_A^{-1}({a_2}^2 S_H^{-1}({a'_2}^1) \rightharpoonup a'_1) \\
&\qquad = \rho_A({a_2}^2 a'_3) a_1({a_2}^1 \rightharpoonup [{a'_2}^2 S_A^{-1}(S_H^{-1}({a'_2}^1) \rightharpoonup a'_1)]) \\
&\qquad = \rho_A({a_2}^2 a'_2) a_1({a_2}^1 \rightharpoonup \epsilon_A(a'_1)1) \\
&\qquad = a_1 \rho_A(a_2 a')
\end{aligned}
$$

To prove the second statement, observe that we have by Proposition 2.10 that:

$$
\begin{aligned}
a^2 {a'_2}^1 \otimes \rho_A(a^3 {a'_2}^2) a_A^R S_A^{-1}(a^1 \rightharpoonup a'_1) &= g_A \otimes \rho_A(a^2 a'_2) a_A^R S_A^{-1}(a^1 \rightharpoonup a'_1) \\
&= g_A \otimes a_1 \rho_A(a_2 a')
\end{aligned}
$$

and therefore we conclude from Proposition 2.12 that:

$$
\begin{aligned}
g_A^{-1} \rightharpoonup a_1 \rho_A(a_2 a') &= S_H^{-1}(a^2 {a'_2}^1) \rightharpoonup (\rho_A(a^3 {a'_2}^2) a_A^R S_A^{-1}(a^1 \rightharpoonup a'_1)) \\
&= \rho_A(a {a'_2}^2) a_A^R S_A^{-1}(S_H^{-1}({a'_2}^1) \rightharpoonup a'_1) \qquad \square
\end{aligned}
$$

The second formula in the above Lemma can be better understood in terms of the categorically coopposite Hopf algebra A^{copp} (cf. [32], subsection 4.3). A^{copp} has the same multiplication, unit and counit as A, but the comultiplication and the antipode of A^{copp} are given by:

$$\Delta_A^{copp} = \sigma_{A,A}^{-1} \circ \Delta_A \qquad S_A^{copp} = S_A^{-1}$$

A^{copp} is not a Yetter-Drinfel'd Hopf algebra, but rather a Hopf algebra in the category where the usual quasisymmetry is replaced by its inverse (cf. [9], sec. 2, p. 33). If we temporarily use the Sweedler notation $\Delta_A^{copp}(a) = a_{(1)} \otimes a_{(2)}$ for the coopposite comultiplication, then the second formula takes the form:

$$\rho_A(a a'_{(1)}) a_A^R S_A^{-1}(a'_{(2)}) = g_A^{-1} \rightharpoonup a_1 \rho_A(a_2 a')$$

We now give the explicit form of the twisted Nakayama automorphisms:

Theorem We have for $a \in A$:

1. $\nu_+(I_A^{-1}(a)) = \alpha_A^R(a_1) S_A^{-2}(a_2)$
2. $\nu_-(I_A(a)) = a_A^L S_A^2(a_1) a_A^R \alpha_A^R(a_2)$

Proof. Using the forms of the Casimir element in Proposition 4.2 and the H-linearity of ν_+, we calculate:

$$\begin{aligned}\nu_+(a) &= S_A^{-1}({\Gamma_A}_2{}^3)\iota_A({\Gamma_A}_2{}^2)\rho_A((S_H^{-1}({\Gamma_A}_2{}^1)\rightharpoonup {\Gamma_A}_1)\nu_+(a))\\ &= S_A^{-1}({\Gamma_A}_2{}^3)(\rho_A \leftharpoonup {\Gamma_A}_2{}^2)((S_H^{-1}({\Gamma_A}_2{}^1)\rightharpoonup {\Gamma_A}_1)\nu_+(a))\\ &= S_A^{-1}({\Gamma_A}_2{}^2)\rho_A({\Gamma_A}_1\nu_+({\Gamma_A}_2{}^1\rightharpoonup a))\\ &= S_A^{-1}({\Gamma_A}_2{}^2)\rho_A(({\Gamma_A}_1{}^1{\Gamma_A}_2{}^1\rightharpoonup a){\Gamma_A}_1{}^2)\end{aligned}$$

We conclude:

$$\begin{aligned}S_A^2(\nu_+(a)) &= \rho_A(({\Gamma_A}^1\rightharpoonup a){\Gamma_A}^2{}_1)S_A({\Gamma_A}^2{}_2)\\ &= \rho_A(({\Gamma_A}^1\rightharpoonup a_1)[({\Gamma_A}^2\rightharpoonup a_2)^1\rightharpoonup {\Gamma_A}^3{}_1])({\Gamma_A}^2\rightharpoonup a_2)^2{\Gamma_A}^3{}_2S_A({\Gamma_A}^3{}_3)\\ &= \rho_A(({\Gamma_A}^1\rightharpoonup a_1)[({\Gamma_A}^2\rightharpoonup a_2)^1\rightharpoonup {\Gamma_A}^3])({\Gamma_A}^2\rightharpoonup a_2)^2\end{aligned}$$

Using Proposition 2.10, this reduces to:

$$\begin{aligned}S_A^2(\nu_+(a)) &= \rho_A((g_A\rightharpoonup a_1)[(g_A\rightharpoonup a_2)^1\rightharpoonup \Gamma_A])(g_A\rightharpoonup a_2)^2\\ &= \iota_A((g_A\rightharpoonup a_2)^1)\rho_A((g_A\rightharpoonup a_1)\Gamma_A)(g_A\rightharpoonup a_2)^2\\ &= \iota_A((g_A\rightharpoonup a_2)^1)\alpha_A^R(g_A\rightharpoonup a_1)(g_A\rightharpoonup a_2)^2\\ &= \iota_A({a_2}^1)\alpha_A^R(a_1)g_A\rightharpoonup {a_2}^2\\ &= \alpha_A^R(a_1)I_A(a_2) = \alpha_A^R(I_A(a_1))I_A(a_2) = \alpha_A^R(I_A(a)_1)I_A(a)_2\end{aligned}$$

In order to prove the second formula, we apply the preceding Lemma:

$$\begin{aligned}\rho_A(a'_2\nu_-(a))a_A^RS_A^{-1}(a'_1) &= \rho_A(a^2(S_H^{-1}(a^1)\rightharpoonup a'_2))a_A^RS_A^{-1}(a'_1)\\ &= \rho_A(a^4(S_H^{-1}(a^1)\rightharpoonup a'_2))a_A^RS_A^{-1}(a^3S_H^{-1}(a^2)\rightharpoonup a'_1)\\ &= {a^2}_1\rho_A({a^2}_2(S_H^{-1}(a^1)\rightharpoonup a'))\end{aligned}$$

Inserting Γ_A for a', we get:

$$\iota_A^{-1}(a^1){a^2}_1\alpha_A^R({a^2}_2) = a_A^RS_A^{-1}({\Gamma_A}_1)\rho_A({\Gamma_A}_2\nu_-(a))$$

But this implies by Proposition 4.2 that:

$$\begin{aligned}&\iota_A^{-1}(a^1)a_A^{R-1}S_A^2(g_A^{-1}\rightharpoonup {a^2}_1)a_A^R\alpha_A^R({a^2}_2)\\ &\qquad = S_A(g_A^{-1}\rightharpoonup {\Gamma_A}_1)a_A^R\rho_A({\Gamma_A}_2\nu_-(a)) = \nu_-(a)\end{aligned}$$

which says that $a_A^LS_A^2(I_A^{-1}(a)_1)a_A^R\alpha_A^R(I_A^{-1}(a)_2) = \nu_-(a)$. □

4.6 It is natural to ask whether the twisted Nakayama automorphisms and the ordinary Nakayama automorphism are interrelated. We can derive an explicit formula for the ordinary Nakayama automorphism in terms of the negative one. It will become clear later that this is not so easy to derive a corresponding formula in the case of the positive one.

Proposition For all $a \in A$, we have: $\nu_A(a) = \iota_A(a^1)\nu_-(\theta_A(a^2))$

Proof. We have:

$$\begin{aligned}
\rho_A(a'\nu_-(a)) &= \rho_A(a^2(S_H^{-1}(a^1) \rightarrow a')) \\
&= \rho_A((S_H^{-1}(a^2)S_H^{-2}(a^3) \rightarrow a^4)(S_H^{-1}(a^1) \rightarrow a')) \\
&= (\rho_A \leftarrow S_H^{-1}(a^1))((S_H^{-2}(a^2) \rightarrow a^3)a') \\
&= \iota_A^{-1}(a^1)\rho_A(\theta_A^{-1}(a^2)a') \\
&= \iota_A^{-1}(a^1)\rho_A(a'\nu_A(\theta_A^{-1}(a^2)))
\end{aligned}$$

By nondegeneracy, this implies $\nu_-(a) = \iota_A^{-1}(a^1)\nu_A(\theta_A^{-1}(a^2))$, which implies the assertion. $\Box$

From the Proposition, we see that the negatively twisted Nakayama automorphism commutes with the ordinary Nakayama automorphism, because it commutes with itself, θ, and, by naturality, with the monoidal transformation $a \mapsto \iota_A(a^1)a^2$. But then, the positively twisted Nakayama automorphism also commutes with the ordinary one, because we have by Proposition 4.4:

$$\begin{aligned}
\rho_A(a\nu_+(\nu_A(a'))) &= \rho_A(\nu_-^{-1}(a)\nu_A(a')) \\
&= \rho_A(a'\nu_-^{-1}(a)) = \rho_A(\nu_A^{-1}(\nu_-^{-1}(a))a') \\
&= \rho_A(\nu_-^{-1}(\nu_A^{-1}(a))a') = \rho_A(\nu_A^{-1}(a)\nu_+(a')) \\
&= \rho_A(\nu_+(a')a) = \rho_A(a\nu_A(\nu_+(a')))
\end{aligned}$$

which implies $\nu_+(\nu_A(a')) = \nu_A(\nu_+(a'))$ by nondegeneracy.

We note that the above Proposition also enables us to calculate the compatibility of the ordinary Nakayama automorphism with the action and the coaction. Using the notation:

$$\varphi : H \rightarrow H,\ h \mapsto S_H^2(g_A^{-1}hg_A) = \iota_A(h_1)S_H^2(h_2)\iota_A^{-1}(h_3)$$

we see that ν_A is an H-linear and colinear map from A to A_φ because it is the composition of the H-linear and colinear mappings:

$$A \xrightarrow{\nu_-} A \xrightarrow{\theta_A} A_{S_H^2} \xrightarrow{\psi_A} A_\varphi$$

where ψ denotes the monoidal transformation $\psi_A(a) = \iota_A(a^1)a^2$.

4.7 In order to formulate our analogue of Radford's formula, we must consider adjoints of linear maps with respect to the bilinear form which is induced by our right integral ρ_A:

Definition If $f : A \to A$ is a K-linear map, denote by $f^* : A \to A$ the unique linear map that satisfies for all $a, a' \in A$:

$$\rho_A(f(a)a') = \rho_A(af^*(a'))$$

f^* will be called the adjoint mapping of f, more precisely the right adjoint mapping.

The following Proposition summarizes the basic properties of the adjunction process:

Proposition Suppose that $f : A \to A$ is a K-linear map.

1. If f is an algebra automorphism which is simultaneously a coalgebra automorphism, then ρ_A is an eigenvector for the transpose of f, that is, there is $\zeta \in K$ such that we have for all $a \in A$:

$$\rho_A(f(a)) = \zeta \rho_A(a)$$

   The adjoint of f is: $f^* = \zeta f^{-1}$

2. Suppose that $\varphi : H \to H$ is a Hopf algebra automorphism that satisfies $\varphi(g_A) = g_A$ and $\iota_A \circ \varphi = \iota_A$. If $f : A \to A_\varphi$ is H-linear and colinear, then f^* is an H-linear and colinear map from A_φ to A.

Proof. We first check that $\rho_A \circ f$ is again a right integral:

$$\rho_A(f(a_1))a_2 = f^{-1}(\rho_A(f(a_1))f(a_2)) = f^{-1}(\rho_A(f(a))1_A) = \rho_A(f(a))1_A$$

Since the space of right integrals is one-dimensional, there is $\zeta \in K$ such that $\rho_A(f(a)) = \zeta \rho_A(a)$. We now have:

$$\rho_A(f(a)f(a')) = \rho_A(f(aa')) = \zeta \rho_A(aa')$$

Inserting $f^{-1}(a')$ instead of a' yields the formula for the adjoint mapping. To prove the second statement, we carry out the following calculation:

$$\begin{aligned}
\rho_A(af^*(h \to a')) &= \rho_A(f(a)(h \to a')) \\
&= \rho_A((h_2 S_H^{-1}(h_1) \to f(a))(h_3 \to a')) \\
&= \iota_A(h_2)\rho_A(f(\varphi^{-1}(S_H^{-1}(h_1)) \to a)a') \\
&= \iota_A(\varphi^{-1}(h_2))\rho_A((\varphi^{-1}(S_H^{-1}(h_1)) \to a)f^*(a')) \\
&= \rho_A((\varphi^{-1}(h_2 S_H^{-1}(h_1)) \to a)(\varphi^{-1}(h_3) \to f^*(a'))) \\
&= \rho_A(a(\varphi^{-1}(h) \to f^*(a')))
\end{aligned}$$

By the nondegeneracy of our bilinear form, we obtain the equation $f^*(h \to a') = \varphi^{-1}(h) \to f^*(a')$. We leave the verification of the colinearity to the reader. $\Box$

4.8 Using Proposition 4.7, it is now easy to calculate the adjoints of the mappings of interest here, because to calculate the adjoint of, for example, the integral transformation, we need only to calculate the eigenvalue of the transpose on ρ_A, which is easy:

Proposition We have:

1. $(M_A^R)^* = M_A^L \qquad (M_A^L)^* = M_A^R \qquad$ (if H is finite-dimensional)
2. $I_A^* = \iota_A(g_A)^2 I_A^{-1}$
3. $(S_A^2 \circ \theta_A)^* = \iota_A(g_A)\alpha_A^R(a_A^R)(S_A^{-2} \circ \theta_A^{-1})$

Proof. Note first that M_A^L is the inverse of M_A^R. Therefore, the first assertion follows from the preceding Proposition if we can show that the transpose of M_A^R takes the eigenvalue 1 on the right integral ρ_A, because M_A^R is an algebra and a coalgebra automorphism by Proposition 3.10. Now we have:

$$\begin{aligned}\rho_A(M_A^R(a)) = \alpha_H^R(a^1)\rho_A(a_H^R \rightharpoonup a^2) &= \alpha_H^R(a^1)\iota_A(a_H^R)\rho_A(a^2)\\ &= \alpha_H^R(g_A)\iota_A(a_H^R)\rho_A(a)\end{aligned}$$

and by Proposition 2.13 we have $\alpha_H^R(g_A)\iota_A(a_H^R) = 1$. The second part of the first statement as well as the second statement follow by a similar reasoning. To prove the last statement, we observe that:

$$\rho_A(\theta_A(a)) = \rho_A(S_H(a^1) \rightharpoonup a^2) = \iota_A^{-1}(a^1)\rho_A(a^2) = \iota_A^{-1}(g_A)\rho_A(a)$$

Since it follows from Corollary 4.2 that the transpose of S_A^2 takes the eigenvalue $\iota_A(g_A)^2\alpha_A^R(a_A^R)$ on ρ_A, we know that the transpose of $S_A^2 \circ \theta_A$ takes the eigenvalue $\iota_A(g_A)\alpha_A^R(a_A^R)$ on ρ_A. Now the assertion follows because we know from Proposition 3.10 that $S_A^2 \circ \theta_A$ is an algebra and a coalgebra automorphism. $\square$

4.9 In the proof of the analogue of Radford's formula, we need to know how the the twisted Nakayama automorphisms and the multiplications with the modular elements commute:

Proposition We have for all $a \in A$:

1. $$\begin{aligned}\nu_+(aa_A^R) &= \alpha_A^R(a_A^R)\nu_+(a)a_A^R \qquad & \nu_+(aa_A^L) &= \alpha_A^R(a_A^L)\nu_+(a)a_A^L\\ \nu_+(a_A^R a) &= \alpha_A^R(a_A^R)a_A^R\nu_+(a) \qquad & \nu_+(a_A^L a) &= \alpha_A^R(a_A^L)a_A^L\nu_+(a)\end{aligned}$$

2. $$\begin{aligned}\nu_-(aa_A^R) &= \alpha_A^R(a_A^R)\nu_-(a)a_A^R \qquad & \nu_-(aa_A^L) &= \alpha_A^R(a_A^L)\nu_-(a)a_A^L\\ \nu_-(a_A^R a) &= \alpha_A^R(a_A^R)a_A^R\nu_-(a) \qquad & \nu_-(a_A^L a) &= \alpha_A^R(a_A^L)a_A^L\nu_-(a)\end{aligned}$$

Proof. Observe first that Proposition 2.12 implies that $I_A(a_A^R) = a_A^R$ and also that $\theta_A(a_A^R) = a_A^R$. Since a_A^R is a grouplike element, we have $S_A^2(a_A^R) = a_A^R$. We can now apply Theorem 4.5 to obtain:

$$\nu_+(a_A^R) = \nu_+(I_A^{-1}(a_A^R)) = \alpha_A^R(a_A^R)S_A^{-2}(a_A^R) = \alpha_A^R(a_A^R)a_A^R$$

The next step is to observe from Proposition 2.12 that we have $\sigma_{A,A}(a \otimes a_A^R) = a_A^R \otimes a$ and $\sigma_{A,A}(a_A^R \otimes a) = a \otimes a_A^R$. This implies by Proposition 4.4 that:

$$\begin{aligned}(\nu_+ \circ \mu_A)(a \otimes a_A^R) &= (\mu_A \circ (\nu_+ \otimes \nu_+) \circ \sigma_{A,A}^{-2})(a \otimes a_A^R) \\ &= (\mu_A \circ (\nu_+ \otimes \nu_+))(a \otimes a_A^R) \\ &= \alpha_A^R(a_A^R)\mu_A(\nu_+(a) \otimes a_A^R)\end{aligned}$$

This proves the first equation. The other formulas can be obtained by a similar reasoning. □

In the proof of the analogue of Radford's formula, we shall need another commutativity result. First, we introduce an abbreviation for the right coregular action of α_A^L:

$$L : A \rightarrow A,\ a \mapsto \alpha_A^L(a_1)a_2$$

Observe that the adjoint L^* of L is, up to scalar multiple, the right coregular action of α_A^R. To see this, we apply α_A^L to the equation in Lemma 4.5 to get:

$$\alpha_A^L(a_1)\rho_A(a_2 a') = \alpha_A^L(a_A^L)\alpha_A^R(a'_1)\rho_A(aa'_2)$$

where we have used the H-linearity of α_A^R from Proposition 2.12. This means that $\rho_A(aL^*(a')) = \rho_A(L(a)a') = \alpha_A^L(a_A^R)\alpha_A^R(a'_1)\rho_A(aa'_2)$, which implies that $L^*(a') = \alpha_A^L(a_A^R)\alpha_A^R(a'_1)a'_2$ by the nondegeneracy of our bilinear form.

Now we can state our commutativity result:

Lemma $L^*(a_A^R\, a) = \alpha_A^R(a_A^R)\, a_A^R L^*(a) \qquad L^*(aa_A^L) = \alpha_A^R(a_A^L)\, L^*(a)a_A^L$

Proof. Since a scalar factor does not matter in the above equations, the first assertion follows from the calculation:

$$\alpha_A^R((a_A^R\, a)_1)(a_A^R\, a)_2 = \alpha_A^R(a_{A1}^R(a_{A2}^{R\ 1} \rightarrow a_1))a_{A2}^{R\ 2}a_2 = \alpha_A^R(a_A^R)\alpha_A^R(a_1)a_A^R a_2$$

The second assertion follows from a similar calculation. □

4.10 We now proceed to prove the main result of this section, an analogue of Radford's formula. We shall need one further technical preparation:

Lemma

1. θ_A^* commutes with θ_A and S_A^2.
2. L^* commutes with I_A.

Proof. To prove the first statement, note that it follows from Proposition 4.7 that we have:

$$\theta_A^*(h \rightarrow a) = S_H^{-2}(h) \rightarrow \theta_A^*(a) \qquad \delta(\theta_A^*(a)) = S_H^{-2}(a^1) \otimes \theta_A^*(a^2)$$

This implies:

$$\theta_A(\theta_A^*(a)) = S_H^{-1}(a^1) \rightarrow \theta_A^*(a^2) = \theta_A^*(\theta_A(a))$$

To prove that θ_A^* also commutes with S_A^2, we argue as follows: Since θ is a natural transformation, θ_A commutes with S_A^2 and therefore with $S_A^2 \circ \theta_A$. Adjoints of commuting maps commute, and therefore we see by Proposition 4.8 that θ_A^* commutes with $S_A^{-2} \circ \theta_A^{-1}$. Since we already know that θ_A and θ_A^* commute, we conclude that θ_A^* commutes with S_A^2. Finally, we have to show that L^* commutes with I_A. From Proposition 2.12 we conclude that $\alpha_A^L \circ I_A = \alpha_A^L$:

$$(\alpha_A^L \circ I_A)(a) = \iota_A(a^1)\alpha_A^L(g_A \rightarrow a^2) = \iota_A(a^1)\alpha_A^L(a^2) = \alpha_A^L(a)$$

This implies $L(I_A(a)) = \alpha_A^L(I_A(a_1))I_A(a_2) = I_A(L(a))$. Therefore L^* commutes with $I_A^* = \iota_A(g_A)^2 I_A^{-1}$, which implies the assertion. □

After all these preparations, the actual proof of the formula is easy:

Theorem For all $a \in A$, we have:

$$(S_A^4 \circ \theta_A^* \circ \theta_A)(a) = \iota_A(g_A^{-1})\alpha_A^R(a_1)a_A^R I_A^2(a_2)a_A^L \alpha_A^L(a_3)$$

Proof. We have from Theorem 4.5 the following formulas for S_A^2:

$$S_A^{-2}(a) = \alpha_A^L(a_1)\nu_+(I_A^{-1}(a_2)) \qquad S_A^2(a) = a_A^R \, \nu_-(I_A(a_1))a_A^L \alpha_A^L(a_2)$$

By Proposition 4.9, the second formula can also be written in the form $S_A^2(a) = \nu_-(a_A^R I_A(a_1)a_A^L)\alpha_A^L(a_2)$. This implies by Proposition 4.4:

$$\begin{aligned}
\rho_A(a(S_A^4 \circ \theta_A^* \circ \theta_A)(a')) &= \rho_A(a(\theta_A^* \circ S_A^2 \circ \theta_A \circ S_A^2)(a')) \\
&= \rho_A(\theta_A(a)(S_A^2 \circ \theta_A \circ S_A^2)(a')) \\
&= \iota_A(g_A)\alpha_A^R(a_A^R)\rho_A(S_A^{-2}(a)S_A^2(a')) \\
&= \iota_A(g_A)\alpha_A^R(a_A^R)\alpha_A^L(a_1)\rho_A(\nu_+(I_A^{-1}(a_2))\nu_-(a_A^R I_A(a'_1)a_A^L))\alpha_A^L(a'_2) \\
&= \iota_A(g_A)\alpha_A^R(a_A^R)\alpha_A^L(a_1)\rho_A(I_A^{-1}(a_2)a_A^R I_A(a'_1)a_A^L)\alpha_A^L(a'_2) \\
&= \iota_A(g_A)\alpha_A^R(a_A^R)\rho_A(I_A^{-1}(L(a))a_A^R I_A(a'_1)a_A^L)\alpha_A^L(a'_2) \\
&= \iota_A(g_A^{-1})\alpha_A^R(a_A^R)\rho_A(L(a)a_A^R I_A^2(a'_1)a_A^L)\alpha_A^L(a'_2) \\
&= \iota_A(g_A^{-1})\alpha_A^R(a'_1)\rho_A(a a_A^R I_A^2(a'_2)a_A^L)\alpha_A^L(a'_3)
\end{aligned}$$

Because the bilinear form $(a, a') \mapsto \rho_A(aa')$ is nondegenerate, the Theorem follows. □

Note that if $H = K$, the base field, then Yetter-Drinfel'd Hopf algebras are the same as ordinary Hopf algebras. In this case, θ_A, θ_A^* and I_A are the identity mappings, and the above formula reduces to:

$$S_A^4(a) = \alpha_A^R(a_1)a_A^R a_2 a_A^L \alpha_A^L(a_3)$$

which is just Radford's well known formula (cf. [22], Prop. 6,p. 347).

4.11 There is one point about the above formula which might appear as not completely satisfactory: It involves the adjoint of θ_A, which is not written down explicitly in terms of the other structure elements and may not be easily accessible. We now proceed to prove a second version of Radford's formula under the additional assumption that also H is finite-dimensional. Since the formula we are going to prove involves the modular transformation, which is not defined if H is infinite-dimensional, it is not even possible to write down the second version of the formula without this assumption. Once again, we need some preparations. Recall the Radford biproduct construction from subsection 3.11. The following Proposition expresses the modular elements and functions of the Radford biproduct in terms of the corresponding elements of A and H:

Proposition Suppose that H is finite-dimensional.

1. The modular functions of the Radford biproduct are given by:

$$\alpha_{A\otimes H}^R(a\otimes h) = \alpha_A^R(a)\iota_A(h_1)\alpha_H^R(h_2) \quad \alpha_{A\otimes H}^L(a\otimes h) = \alpha_A^L(a)\alpha_H^L(h_1)\iota_A^{-1}(h_2)$$

2. The modular elements of the Radford biproduct are given by:

$$a_{A\otimes H}^R = a_A^R \otimes g_A a_H^R \qquad a_{A\otimes H}^L = a_A^L \otimes g_A^{-1} a_H^L$$

Proof. We know from [23], Prop. 3, p. 333 that $\Lambda_A \otimes \Lambda_{H1}\iota_A(\Lambda_{H2})$ is a nonzero left integral of $A \otimes H$ if $\Lambda_A \in A$ and $\Lambda_H \in H$ are nonzero left integrals. Using Proposition 2.12, we now calculate:

$$\begin{aligned}(\Lambda_A \otimes \Lambda_{H1}\iota_A(\Lambda_{H2}))(a \otimes h) &= \Lambda_A(\Lambda_{H1} \rightharpoonup a) \otimes \Lambda_{H2}h\iota_A(\Lambda_{H3}) \\ &= \alpha_A^L(\Lambda_{H1} \rightharpoonup a)\Lambda_A \otimes \Lambda_{H2}h\iota_A(\Lambda_{H3}) \\ &= \alpha_A^L(a)\Lambda_A \otimes \Lambda_{H1}h\iota_A(\Lambda_{H2}) \\ &= \alpha_A^L(a)\Lambda_A \otimes \Lambda_{H1}h_1\iota_A(\Lambda_{H2}h_2)\iota_A^{-1}(h_3) \\ &= \alpha_A^L(a)\Lambda_A \otimes \Lambda_{H1}\iota_A(\Lambda_{H2})\alpha_H^L(h_1)\iota_A^{-1}(h_2)\end{aligned}$$

This establishes the form of the left modular function. The right modular function may be derived by a similar calculation using the fact that $\Gamma_A \otimes \Gamma_H$ is a nonzero right integral of the Radford biproduct if $\Gamma_A \in A$ and $\Gamma_H \in H$ are nonzero right integrals, which we know from [23], Prop. 3 or by straightforward verification. Now we turn to the left modular element. Select nonzero left integrals $\lambda_A \in A^*$ and $\lambda_H \in H^*$. Then we know from [23], Prop. 4 that the function

$$\lambda_{A\otimes H}(a \otimes h) = \lambda_A(a)\lambda_H(g_A h)$$

is a nonzero left integral in $(A \otimes H)^*$. From this, we obtain:

$$\begin{aligned}\lambda_{A\otimes H}((a \otimes h)_1)(a \otimes h)_2 &= \lambda_A(a_1)\lambda_H(g_A {a_2}^1 h_1){a_2}^2 \otimes h_2 \\ &= \lambda_A(a)\lambda_H(g_A h_1)a_A^L \otimes h_2 \\ &= \lambda_A(a)\lambda_H(g_A h_1)a_A^L \otimes g_A^{-1} g_A h_2 \\ &= \lambda_A(a)\lambda_H(g_A h)a_A^L \otimes g_A^{-1} a_H^L\end{aligned}$$

This establishes the form of the left modular element. The form of the right modular element can be established by a similar calculation using the fact that $\rho_A \otimes \rho_H \in (A \otimes H)^*$ is a nonzero right integral if $\rho_A \in A^*$ and $\rho_H \in H^*$ are nonzero right integrals. □

We note that a similar formula has appeared earlier in [5], Remark 5.9.

We now derive the finite-dimensional form of the formula:

Theorem We have for all $a \in A$:

$$(S_A^2 \circ \theta_A)^2(a) = \alpha_A^R(a_1)a_A^R I_A(M_A^R(a_2))a_A^L \alpha_A^L(a_3)$$

Proof. By Radford's formula for ordinary Hopf algebras we have:

$$S_{A\otimes H}^4(a \otimes 1) = \alpha_{A\otimes H}^R((a \otimes 1)_1)a_{A\otimes H}^R(a \otimes 1)_2 a_{A\otimes H}^L \alpha_{A\otimes H}^L((a \otimes 1)_3)$$

We can calculate the left hand side by Proposition 3.11 and the right hand side by the preceding Proposition. Using the formula

$$\Delta_{A\otimes H}^2(a \otimes 1) = (a_1 \otimes {a_2}^1 {a_3}^1) \otimes ({a_2}^2 \otimes {a_3}^2) \otimes ({a_3}^3 \otimes 1)$$

for the twofold coproduct, we arrive at:

$$\begin{aligned}&S_A^4(\theta_A^2(a)) \otimes 1 \\ &= \alpha_A^R(a_1)\iota_A({a_2}^1 {a_3}^1)\alpha_H^R({a_2}^2 {a_3}^2)(a_A^R \otimes g_A a_H^R)({a_2}^3 \otimes {a_3}^3)(a_A^L \otimes g_A^{-1} a_H^L)\alpha_A^L({a_3}^4) \\ &= \alpha_A^R(a_1)\iota_A({a_2}^1)\alpha_H^R({a_2}^2)(a_A^R \otimes g_A a_H^R)({a_2}^3 \otimes 1)(a_A^L \otimes g_A^{-1} a_H^L)\alpha_A^L(a_3) \\ &= \alpha_A^R(a_1)\iota_A({a_2}^1)\alpha_H^R({a_2}^2)(a_A^R(g_A a_H^R \rightarrow {a_2}^3)a_A^L \otimes 1)\alpha_A^L(a_3) \\ &= \alpha_A^R(a_1)(a_A^R I_A(M_A^R(a_2))a_A^L \otimes 1)\alpha_A^L(a_3)\end{aligned}$$

The consideration of the A-component yields the assertion. □

4.12 A comparison between both variants of Radford's formulas that we have proved so far yields interesting interrelations between the natural transformations that we are considering.

Proposition Suppose that H is finite-dimensional. Then we have:

1. $\rho_A(\theta_A(M^L_A(a))a') = \iota_A(g_A^{-1})\rho_A(a\theta_A(I_A(a')))$
2. $\theta^*_A = \iota_A(g_A^{-1})\nu_+ \circ \theta_A^{-1} \circ \nu_-^{-1} = \iota_A(g_A^{-1})\theta_A \circ M^L_A \circ I_A$

Proof. By comparing the two formulas for S^4_A in Theorem 4.10 and Theorem 4.11, we see that:

$$\begin{aligned}(\theta^*_A \circ \theta_A^{-1})(\alpha^R_A(a_1)a^R_A I_A(M^R_A(a_2))a^L_A)\alpha^L_A(a_3)\\ = \iota_A(g_A^{-1})\alpha^R_A(a_1)a^R_A I^2_A(a_2)a^L_A\alpha^L_A(a_3)\end{aligned}$$

Inserting $\alpha^L_A(a_1)a_2\alpha^R_A(a_3)$ in place of a, we get:

$$(\theta^*_A \circ \theta_A^{-1})(a^R_A I_A(M^R_A(a))a^L_A) = \iota_A(g_A^{-1})a^R_A I^2_A(a)a^L_A$$

Since I_A and M^R_A are algebra homomorphisms by Proposition 3.10, it follows easily from Proposition 2.12 that we have for example $I_A(a^L_A a) = I_A(a^L_A)I_A(a) = a^L_A I_A(a)$. Therefore, inserting $a^L_A a a^R_A$ for a, we get $(\theta^*_A \circ \theta_A^{-1})(I_A(M^R_A(a))) = \iota_A(g_A^{-1})I^2_A(a)$. It is easy to deduce from Theorem 2.11 and Proposition 3.6 that I_A commutes with M^R_A and, since θ_A commutes with θ^*_A by Lemma 4.10, we get: $\theta^*_A \circ M^R_A = \iota_A(g_A^{-1})\theta_A \circ I_A$. This implies:

$$\rho_A(\theta_A(a)M^R_A(a')) = \iota_A(g_A^{-1})\rho_A(a\theta_A(I_A(a')))$$

which implies the first assertion by Proposition 4.8.

To prove the second assertion, we observe that, since θ is a ribbon transformation, we have as in subsection 3.10:

$$\theta_A \circ \mu_A = \mu_A \circ \theta_{A\otimes A} = \mu_A \circ (\theta_A \otimes \theta_A) \circ \sigma_{A,A}^{-2}$$

where $\mu_A : A \otimes A \to A$ denotes the multiplication map. Since we know from the proof of Proposition 4.8 that we have $\rho_A \circ \theta_A = \iota_A(g_A^{-1})\rho_A$, we have:

$$\rho_A \circ \mu_A \circ \sigma_{A,A}^{-1} \circ (\theta_A \otimes \theta_A) = \iota_A(g_A^{-1})\rho_A \circ \mu_A \circ \sigma_{A,A}$$

But this implies by the definition of the twisted Nakayama automorphisms that we have:

$$\rho_A \circ \mu_A \circ (id_A \otimes \nu_-) \circ (\theta_A \otimes \theta_A) = \iota_A(g_A^{-1})\rho_A \circ \mu_A \circ (id_A \otimes \nu_+)$$

Since the left hand side is also $\rho_A \circ \mu_A \circ (id_A \otimes \theta^*_A) \circ (id_A \otimes \nu_-) \circ (id_A \otimes \theta_A)$ by the definition of adjoint maps, we get by nondegeneracy that:

$$\iota_A(g_A^{-1})\nu_+ = \theta^*_A \circ \nu_- \circ \theta_A$$

This implies the first equality. We have already seen above a variant of the second equality. □

5 Integrals and triangular decompositions

5.1 In this section, A (resp. B) is a fixed left (resp. right) Yetter-Drinfel'd bialgebra over a bialgebra H. Δ_A (resp. Δ_B) and ϵ_A (resp. ϵ_B) denote the comultiplication and the counit. Recall the first construction of [31]:

Definition
A pair (A, B) consisting of a left and a right Yetter-Drinfel'd bialgebra together with linear mappings $\rightharpoonup: B \otimes A \to A$, $\leftharpoonup: B \otimes A \to B$ and $\sharp : B \otimes A \to H$ is called a Yetter-Drinfel'd bialgebra pair if:

1. A is a left B-module via $\rightharpoonup$.
2. B is a right A-module via $\leftharpoonup$.

and the following compatibility conditions are satisfied:

1. $\Delta_A(b \rightharpoonup a) = ({b_1}^1 \rightharpoonup a_1) \otimes ({b_1}^2 \to (b_2 \rightharpoonup a_2))$
   $\Delta_B(b \leftharpoonup a) = ((b_1 \leftharpoonup a_1) \leftarrow {a_2}^1) \otimes (b_2 \leftharpoonup {a_2}^2)$
2. $\Delta_H(b\sharp a) = ({b_1}^1\sharp a_1){a_2}^1 \otimes {b_1}^2(b_2\sharp {a_2}^2)$
3. $b \rightharpoonup (aa') = ({b_1}^1 \rightharpoonup a_1)({b_1}^2(b_2\sharp a_2){a_3}^1 \to [(b_3 \leftharpoonup {a_3}^2) \rightharpoonup a'])$
   $(bb') \leftharpoonup a = ([b \leftharpoonup ({b'_1}^1 \rightharpoonup a_1)] \leftarrow {b'_1}^2(b'_2\sharp a_2){a_3}^1)(b'_3 \leftharpoonup {a_3}^2)$
4. $b\sharp(aa') = (b_1\sharp a_1){a_2}^1((b_2 \leftharpoonup {a_2}^2)\sharp a')$
   $(bb')\sharp a = (b\sharp({b'_1}^1 \rightharpoonup a_1)){b'_1}^2(b'_2\sharp a_2)$
5. $\epsilon_H(b\sharp a) = \epsilon_A(a)\epsilon_B(b)$
6. $b \rightharpoonup 1 = \epsilon_B(b)1,\ 1 \leftharpoonup a = \epsilon_A(a)1$
7. $b\sharp 1 = \epsilon_B(b)1,\ 1\sharp a = \epsilon_A(a)1$
8. $({b_1}^1 \rightharpoonup a_1)^1 {b_1}^2(b_2\sharp a_2) \otimes ({b_1}^1 \rightharpoonup a_1)^2 = ({b_1}^1\sharp a_1){a_2}^1 \otimes ({b_1}^2 \to (b_2 \rightharpoonup {a_2}^2))$
   $(b_2 \leftharpoonup {a_2}^2)^1 \otimes (b_1\sharp a_1){a_2}^1(b_2 \leftharpoonup {a_2}^2)^2 = (({b_1}^1 \leftharpoonup a_1) \leftarrow {a_2}^1) \otimes {b_1}^2(b_2\sharp {a_2}^2)$
9. $b \rightharpoonup (h \to a) = h_1 \to ((b \leftarrow h_2) \rightharpoonup a)$
   $(b \leftarrow h) \leftharpoonup a = (b \leftharpoonup (h_1 \to a)) \leftarrow h_2$
10. $(b\sharp(h_1 \to a))h_2 = h_1((b \leftarrow h_2)\sharp a)$
11. $(b_1 \rightharpoonup a_1) \otimes (b_2 \leftharpoonup a_2) = ({b_1}^2 \to (b_2 \rightharpoonup {a_2}^2)) \otimes (({b_1}^1 \leftharpoonup a_1) \leftarrow {a_2}^1)$

These conditions are of course required for all $a,\ a' \in A$, $b,\ b' \in B$ and $h \in H$.

In this situation, we can carry out the first construction:

Theorem Given a Yetter-Drinfel'd bialgebra pair, $A \otimes H \otimes B$ is a bialgebra with multiplication

$$\mu : (A \otimes H \otimes B) \otimes (A \otimes H \otimes B) \to A \otimes H \otimes B$$
$$(a \otimes h \otimes b) \otimes (a' \otimes h' \otimes b') \mapsto$$
$$a(h_1 \rightarrow ({b_1}^1 \rightharpoonup a'_1)) \otimes h_2 {b_1}^2 (b_2 \sharp a'_2) {a'_3}^1 h'_1 \otimes ((b_3 \leftharpoonup {a'_3}^2) \leftarrow h'_2) b'$$

unit element $1 \otimes 1 \otimes 1$, comultiplication

$$\Delta : A \otimes H \otimes B \to (A \otimes H \otimes B) \otimes (A \otimes H \otimes B)$$
$$a \otimes h \otimes b \mapsto (a_1 \otimes {a_2}^1 h_1 \otimes {b_1}^1) \otimes ({a_2}^2 \otimes h_2 {b_1}^2 \otimes b_2)$$

and counit

$$\epsilon : A \otimes H \otimes B \to K$$
$$a \otimes h \otimes b \mapsto \epsilon_A(a) \epsilon_H(h) \epsilon_B(b)$$

If A, H and B possess antipodes, then $A \otimes H \otimes B$ has the antipode

$$S(a \otimes h \otimes b) = (1 \otimes 1 \otimes S_B(b^1))(1 \otimes S_H(a^1 h b^2) \otimes 1)(S_A(a^2) \otimes 1 \otimes 1)$$

The formulas for the multiplication and the comultiplication in this construction are strictly symmetric. This can be formalized as follows: Consider $B^{op\ cop}$ as a left Yetter-Drinfel'd bialgebra over $H^{op\ cop}$ and $A^{op\ cop}$ as a right Yetter-Drinfel'd bialgebra over $H^{op\ cop}$ as in subsection 2.5. Introduce a left $A^{op\ cop}$-action on $B^{op\ cop}$ via

$$A^{op\ cop} \otimes B^{op\ cop} \to B^{op\ cop}, a \otimes b \mapsto (b \leftharpoonup a)$$

Introduce a right $B^{op\ cop}$-action on $A^{op\ cop}$ via

$$a \otimes b \mapsto (b \rightharpoonup a)$$

Similarly, we can introduce a mapping

$$A^{op\ cop} \otimes B^{op\ cop} \to H^{op\ cop}, a \otimes b \mapsto b \sharp a$$

Using these structure elements, we can form $B^{op\ cop} \otimes H^{op\ cop} \otimes A^{op\ cop}$.

Lemma The map

$$B^{op\ cop} \otimes H^{op\ cop} \otimes A^{op\ cop} \to (A \otimes H \otimes B)^{op\ cop}$$
$$b \otimes h \otimes a \mapsto a \otimes h \otimes b$$

is a bialgebra isomorphism.

Proof. This is clear from the definitions. Observe that it is not even necessary to check the compatibility conditions above since by [32], subsection 3.10 we know that these are also necessary conditions for $B^{op\ cop} \otimes H^{op\ cop} \otimes A^{op\ cop}$ being a bialgebra, which is clear since it is isomorphic to $(A \otimes H \otimes B)^{op\ cop}$. $\square$

5.2 We shall assume for the rest of this section that A, H and B are finite-dimensional and do have antipodes S_A, S_H and S_B respectively. The purpose of this subsection is to prove the following theorem:

Theorem

1. Suppose that Λ_A, Λ_H and Λ_B are nonzero left integrals of A, H and B respectively. Suppose that ρ_A is a right integral of A^* satisfying $\rho_A(\Lambda_A) = 1$. Set:
$$\omega_B : B \to K, b \mapsto \rho_A(S_B(b^1) \rightharpoonup \Lambda_A)\iota_A^{-1}(b^2)$$
where ι_A is the integral character of A. Then ω_B is an H-linear character of B and
$$\Lambda_A \otimes \Lambda_{H1}\iota_A(\Lambda_{H2}) \otimes \Lambda_{B1}\omega_B^{-1}(\Lambda_{B2})$$
is a nonzero left integral of $A \otimes H \otimes B$.

2. Suppose that Γ_A, Γ_H and Γ_B are nonzero right integrals of A, H and B respectively. Suppose that λ_B is a left integral of B^* satisfying $\lambda_B(\Gamma_B) = 1$. Set:
$$\omega_A : A \to K, a \mapsto \iota_B^{-1}(a^1)\lambda_B(\Gamma_B \leftharpoonup S_A(a^2))$$
where ι_B is the integral character of B. Then ω_A is an H-linear character of A and
$$\omega_A^{-1}(\Gamma_{A1})\Gamma_{A2} \otimes \iota_B(\Gamma_{H1})\Gamma_{H2} \otimes \Gamma_B$$
is a nonzero right integral of $A \otimes H \otimes B$.

Proof. The second part is the dualization of the first part via Lemma 5.1. It remains to show the first part. This proceeds in several steps.

(1) We show first that ω_B is H-linear. By condition (9) in Definition 5.1, we have:

$$S_H(h_1) \to (b \rightharpoonup (h_2 \to a)) = S_H(h_1)h_2 \to ((b \leftarrow h_3) \rightharpoonup a) = (b \leftarrow h) \rightharpoonup a$$

We now calculate, using the Yetter-Drinfel'd condition and Proposition 2.10:

$$\begin{aligned}
\omega_B(b \leftarrow h) &= \rho_A(S_B((b \leftarrow h)^1) \rightharpoonup \Lambda_A)\iota_A^{-1}((b \leftarrow h)^2) \\
&= \rho_A(S_B(b^1 \leftarrow h_2) \rightharpoonup \Lambda_A)\iota_A^{-1}(S_H(h_1)b^2h_3) \\
&= \rho_A(S_H(h_2) \to (S_B(b^1) \rightharpoonup (h_3 \to \Lambda_A)))\iota_A(h_1)\iota_A^{-1}(b^2)\iota_A^{-1}(h_4) \\
&= \iota_A(S_H(h_2))\rho_A(S_B(b^1) \rightharpoonup \iota_A(h_3)\Lambda_A)\iota_A(h_1)\iota_A^{-1}(b^2)\iota_A^{-1}(h_4) \\
&= \epsilon_H(h)\omega_B(b)
\end{aligned}$$

(2) By [19], Theorem 2.1.3, p. 18 we know that $A \otimes H \otimes B$ contains a nonzero integral Λ. Write:
$$\Lambda = \sum_{i=1}^{k} a_{(i)} \otimes h_{(i)} \otimes b_{(i)}$$

where k is minimal so that $a_{(1)} \otimes h_{(1)}, \dots, a_{(k)} \otimes h_{(k)}$ and $b_{(1)}, \dots, b_{(k)}$ are linearly independent. We have for $a \in A$, $h \in H$:

$$\sum_{i=1}^{k} (a \otimes h)(a_{(i)} \otimes h_{(i)}) \otimes b_{(i)} = (a \otimes h \otimes 1)\Lambda = \epsilon_A(a)\epsilon_H(h)\Lambda$$
$$= \sum_{i=1}^{k} \epsilon_A(a)\epsilon_H(h)(a_{(i)} \otimes h_{(i)}) \otimes b_{(i)}$$

and therefore, for all $i = 1, \dots, k$, $a_{(i)} \otimes h_{(i)}$ is a left integral in the Radford biproduct $A \otimes H$. We know from [23],Prop. 3, p. 333 that $\Lambda_A \otimes \Lambda_{H\,1}\iota_A(\Lambda_{H\,2})$ is a left integral of $A \otimes H$ and therefore we have, for all $i = 1, \dots, k$:

$$a_{(i)} \otimes h_{(i)} = \zeta_i \Lambda_A \otimes \Lambda_{H\,1}\iota_A(\Lambda_{H\,2})$$

for some $\zeta_i \in K$. If we set $x_B = \sum_{i=1}^{k} \zeta_i b_{(i)}$, we see that:

$$\Lambda = \Lambda_A \otimes \Lambda_{H\,1}\iota_A(\Lambda_{H\,2}) \otimes x_B$$

(3) We denote by ω''_B the restriction of the right modular function of $A \otimes H \otimes B$ to B:

$$\omega''_B(b) := \alpha^R_{A \otimes H \otimes B}(1 \otimes 1 \otimes b)$$

We have:

$$\Lambda_A \otimes \Lambda_{H\,1}\iota_A(\Lambda_{H\,2}) \otimes x_B b = \Lambda(1 \otimes 1 \otimes b) = \Lambda_A \otimes \Lambda_{H\,1}\iota_A(\Lambda_{H\,2}) \otimes x_B \omega''_B(b)$$

and therefore we conclude that $x_B b = \omega''_B(b) x_B$. So ω''_B is an augmentation and x_B is the corresponding integral. Dualizing Proposition 2.10 with the help of Lemma 2.5, we see that B is a Frobenius algebra and therefore we know by Proposition 2.7 that the space of such integrals is one-dimensional. By the reasoning we have already encountered several times we conclude that we have

$$b x_B = \omega'_B(b) x_B$$

for some character ω'_B of B.

(4) Let us now do a little calculation. We have:

$$\Lambda_A \otimes \Lambda_{H\,1}\iota_A(\Lambda_{H\,2}) \otimes \epsilon_B(b) x_B = (1 \otimes 1 \otimes b)\Lambda_A \otimes \Lambda_{H\,1}\iota_A(\Lambda_{H\,2}) \otimes x_B$$
$$= ({b_1}^1 \rightharpoonup \Lambda_{A\,1}) \otimes {b_1}^2(b_2 \sharp \Lambda_{A\,2}){\Lambda_{A\,3}}^1 \Lambda_{H\,1} \otimes ((b_3 \leftharpoonup {\Lambda_{A\,3}}^2) \leftarrow \Lambda_{H\,2}) x_B \iota_A(\Lambda_{H\,3})$$
$$= ({b_1}^1 \rightharpoonup \Lambda_{A\,1}) \otimes {b_1}^2(b_2 \sharp \Lambda_{A\,2}){\Lambda_{A\,3}}^1 \Lambda_{H\,1} \otimes \omega'_B((b_3 \leftharpoonup {\Lambda_{A\,3}}^2) \leftarrow \Lambda_{H\,2}) \iota_A(\Lambda_{H\,3}) x_B$$

and can conclude that

$$\Lambda_A \otimes \Lambda_{H\,1}\iota_A(\Lambda_{H\,2})\epsilon_B(b) =$$
$$({b_1}^1 \rightharpoonup \Lambda_{A\,1}) \otimes {b_1}^2(b_2 \sharp \Lambda_{A\,2}){\Lambda_{A\,3}}^1 \Lambda_{H\,1}\omega'_B((b_3 \leftharpoonup {\Lambda_{A\,3}}^2) \leftarrow \Lambda_{H\,2})\iota_A(\Lambda_{H\,3})$$

This in turn implies:

$$\begin{aligned}
&S_B(b^1) \rightharpoonup \Lambda_A \otimes b^2 \Lambda_{H1} \iota_A(\Lambda_{H2}) \\
&= S_B({b_1}^1) \rightharpoonup \epsilon_B(b_2)\Lambda_A \otimes {b_1}^2 \Lambda_{H1} \iota_A(\Lambda_{H2}) \\
&= S_B({b_1}^1){b_2}^1 \rightharpoonup \Lambda_{A1} \otimes \\
&\quad {b_1}^2 {b_2}^2 (b_3 \sharp \Lambda_{A2}) {\Lambda_{A3}}^1 \Lambda_{H1} \iota_A(\Lambda_{H3}) \omega'_B((b_4 \leftharpoonup {\Lambda_{A3}}^2) \leftharpoonup \Lambda_{H2}) \\
&= S_B({b_1}^1{}_1){b_1}^1{}_2 \rightharpoonup \Lambda_{A1} \otimes \\
&\quad {b_1}^2 (b_2 \sharp \Lambda_{A2}) {\Lambda_{A3}}^1 \Lambda_{H1} \iota_A(\Lambda_{H3}) \omega'_B((b_3 \leftharpoonup {\Lambda_{A3}}^2) \leftharpoonup \Lambda_{H2}) \\
&= \Lambda_{A1} \otimes (b_1 \sharp \Lambda_{A2}) {\Lambda_{A3}}^1 \Lambda_{H1} \iota_A(\Lambda_{H3}) \omega'_B((b_2 \leftharpoonup {\Lambda_{A3}}^2) \leftharpoonup \Lambda_{H2})
\end{aligned}$$

Now recall that by Proposition 2.10 we have:

$$\rho_A(a\Lambda_{A1})\Lambda_{A2} = \iota_A(a^1) S_A^{-1}(a^2)$$

Thus we see that:

$$\begin{aligned}
&\rho_A(a(S_B(b^1) \rightharpoonup \Lambda_A)) b^2 \Lambda_{H1} \iota_A(\Lambda_{H2}) \\
&= \rho_A(a\Lambda_{A1})(b_1 \sharp \Lambda_{A2}) {\Lambda_{A3}}^1 \Lambda_{H1} \iota_A(\Lambda_{H3}) \omega'_B((b_2 \leftharpoonup {\Lambda_{A3}}^2) \leftharpoonup \Lambda_{H2}) \\
&= \iota_A(a^1)(b_1 \sharp S_A^{-1}(a^2)_1) S_A^{-1}(a^2)_2{}^1 \Lambda_{H1} \iota_A(\Lambda_{H3}) \omega'_B((b_2 \leftharpoonup S_A^{-1}(a^2)_2{}^2) \leftharpoonup \Lambda_{H2})
\end{aligned}$$

Proposition 2.10 implies for the ordinary Hopf algebra H that $\Lambda_{H1} \otimes S_H(\Lambda_{H2})$ is a Casimir element. Therefore we have that $h\Lambda_{H1}\iota_A(\Lambda_{H2}) = \Lambda_{H1}\iota_A(S_H^{-1}(h)\Lambda_{H2})$ $= \iota_A^{-1}(h)\Lambda_{H1}\iota_A(\Lambda_{H2})$. Now, inserting $a = 1$ into the above equation yields:

$$\rho_A(S_B(b^1) \rightharpoonup \Lambda_A)\iota_A^{-1}(b^2)\Lambda_{H1}\iota_A(\Lambda_{H2}) = \Lambda_{H1}\iota_A(\Lambda_{H3})\omega'_B(b \leftharpoonup \Lambda_{H2})$$

By applying Proposition 2.10 to H^{cop} instead of A, we see that H is a Frobenius algebra with Frobenius morphism λ_H, a left integral of H^* satisfying $\lambda_H(\Lambda_H) =$ 1, and corresponding Casimir element $\Lambda_{H2} \otimes S_H^{-1}(\Lambda_{H1})$. As above, we therefore have for all $h \in H$:

$$h = \lambda_H(S_H^{-1}(\Lambda_{H1})h)\Lambda_{H2}$$

Using this and the above equation, we see:

$$\begin{aligned}
&\rho_A(S_B(b^1) \rightharpoonup \Lambda_A)\iota_A^{-1}(b^2)\iota_A(h) \\
&\qquad = \rho_A(S_B(b^1) \rightharpoonup \Lambda_A)\iota_A^{-1}(b^2)\lambda_H(S_H^{-1}(\Lambda_{H1})h)\iota_A(\Lambda_{H2}) \\
&\qquad = \lambda_H(S_H^{-1}(\Lambda_{H1})h)\iota_A(\Lambda_{H3})\omega'_B(b \leftharpoonup \Lambda_{H2}) \\
&\qquad = \iota_A(h_2)\omega'_B(b \leftharpoonup h_1)
\end{aligned}$$

Inserting $h = 1$ into this equation we get:

$$\omega'_B(b) = \rho_A(S_B(b^1) \rightharpoonup \Lambda_A)\iota_A^{-1}(b^2)$$

That is, we finally conclude that ω_B and ω'_B are equal.

(5) Since we have already seen that ω_B is H-linear, we know from subsection 2.6 dualized via Lemma 2.5 that $\phi : B \to B, b \mapsto b_1 \omega_B^{-1}(b_2)$ is an algebra isomorphism. Therefore, we have $\phi(b)\phi(\Lambda_B) = \epsilon_B(b)\phi(\Lambda_B)$ and therefore $\phi(\Lambda_B) = \Lambda_{B1}\omega_B^{-1}(\Lambda_{B2})$ is a nonzero left integral with respect to the character ω_B. But now by Proposition 2.7, we know that the space of such integrals is one-dimensional, and therefore x_B must be proportional to $\Lambda_{B1}\omega_B^{-1}(\Lambda_{B2})$. This finishes the proof of the first part of the Proposition. □

It should be noted that the assumption that A, H and B is necessary, since by [34], Cor. 2.7, p. 330 or [33], Chap. V, Ex. 4, p. 108 an infinite-dimensional Hopf algebra does not contain a nonzero integral.

5.3 As explained in [31], the construction discussed above contains as a special case the so-called double crossproduct (cf. [35]). Here, A and B are ordinary Hopf algebras acting on each other via

$$\rightharpoonup : B \otimes A \to A \qquad \leftharpoonup : B \otimes A \to B$$

such that the following conditions are satisfied:

1. $\Delta_A(b \rightharpoonup a) = (b_1 \rightharpoonup a_1) \otimes (b_2 \rightharpoonup a_2)$
   $\Delta_B(b \leftharpoonup a) = (b_1 \leftharpoonup a_1) \otimes (b_2 \leftharpoonup a_2)$
2. $b \rightharpoonup (aa') = (b_1 \rightharpoonup a_1)((b_2 \leftharpoonup a_2) \rightharpoonup a')$
   $(bb') \leftharpoonup a = (b \leftharpoonup (b'_1 \rightharpoonup a_1))(b'_2 \leftharpoonup a_2)$
3. $\epsilon_A(b \rightharpoonup a) = \epsilon_B(b)\epsilon_A(a) = \epsilon_B(b \leftharpoonup a)$
4. $b \rightharpoonup 1 = \epsilon_B(b)1, 1 \leftharpoonup a = \epsilon_A(a)1$
5. $(b_1 \rightharpoonup a_1) \otimes (b_2 \leftharpoonup a_2) = (b_2 \rightharpoonup a_2) \otimes (b_1 \leftharpoonup a_1)$

In this situation, $A \otimes B$ is a Hopf algebra with multiplication

$$(a \otimes b)(a' \otimes b') = a(b_1 \rightharpoonup a'_1) \otimes (b_2 \leftharpoonup a'_2)b'$$

unit element $1 \otimes 1$, comultiplication

$$\Delta(a \otimes b) = (a_1 \otimes b_1) \otimes (a_2 \otimes b_2)$$

counit

$$\epsilon(a \otimes b) = \epsilon_A(a)\epsilon_B(b)$$

and antipode

$$S(a \otimes b) = (1 \otimes S_B(b))(S_A(a) \otimes 1)$$

As a corollary, we can calculate the integrals of the double crossproduct:

Corollary
Suppose that A and B are finite-dimensional.

1. Suppose that Λ_A resp. Λ_B are nonzero left integrals of A resp. B. Choose a right integral ρ_A of A^* satisfying $\rho_A(\Lambda_A) = 1$ and define
$$\omega_B : B \to K,\ b \mapsto \rho_A(S_B(b) \rightharpoonup \Lambda_A)$$
Then ω_B is a character of B and $\Lambda_A \otimes \Lambda_{B1}\omega_B^{-1}(\Lambda_{B2})$ is a nonzero left integral of the double crossproduct $A \otimes B$.

2. Suppose that Γ_A resp. Γ_B are nonzero right integrals of A resp. B. Choose a left integral λ_B of B^* satisfying $\lambda_B(\Gamma_B) = 1$ and define
$$\omega_A : A \to K,\ a \mapsto \lambda_B(\Gamma_B \leftharpoonup S_A(a))$$
Then ω_A is a character of A and
$$\omega_A^{-1}(\Gamma_{A1})\Gamma_{A2} \otimes \Gamma_B$$
is a nonzero right integral of the double crossproduct $A \otimes B$.

5.4 In this subsection we calculate the integrals of the second construction given in [31]. We work in the following situation: H is a finite-dimensional commutative and cocommutative Hopf algebra. We assume that A is a finite-dimensional left Yetter-Drinfel'd Hopf algebra over H. We impose the following main assumption on A:
$$\forall a, a' \in A : (a^1 \rightharpoonup a') \otimes a^2 = a'^2 \otimes (a'^1 \rightharpoonup a)$$
We introduce a right Yetter-Drinfel'd Hopf algebra B in the following way: We set $B = A^*$ as an algebra and as an H-module in the sense of subsection 2.3. If δ_{A^*} and Δ_{A^*} denote the cooperation and comultiplication from subsection 2.3 respectively, we define the cooperation and the comultiplication of B by:
$$\delta_B = (id_{A^*} \otimes S_H) \circ \delta_{A^*} \qquad \Delta_B = \sigma^{-1}_{A^*,A^*} \circ \Delta_{A^*}$$
where σ is the braiding in the category of right Yetter-Drinfel'd modules. When we use Sweedler's notation, we refer to the structure of B and not to the structure of A^*.

Let $\langle \cdot, \cdot \rangle_A : A \otimes B \to K, a \otimes b \mapsto b(a)$ denote the natural pairing between a vector space and its dual. We also use the form $\langle \cdot, \cdot \rangle_B : A \otimes B \to K, a \otimes b \mapsto \langle S_A^{-1}(a), b \rangle_A$.

Recall the second construction from [31]:

Proposition
$A \otimes H \otimes B$ is a Hopf algebra with comultiplication:

$$\Delta : A \otimes H \otimes B \to (A \otimes H \otimes B) \otimes (A \otimes H \otimes B)$$

$$a \otimes h \otimes b \mapsto (a_1 \otimes {a_2}^1 h_1 \otimes {b_1}^1) \otimes ({a_2}^2 \otimes h_2 {b_1}^2 \otimes b_2)$$

multiplication:

$$\begin{aligned}\mu : (A \otimes H \otimes B) \otimes (A \otimes H \otimes B) &\to A \otimes H \otimes B \\ (a \otimes h \otimes b) \otimes (a' \otimes h' \otimes b') &\mapsto \\ a(h_1 \to ({b_1}^1 \rightharpoonup a'_1)) \otimes h_2 {b_1}^2 (b_2 \sharp a'_2) {a'_3}^1 h'_1 &\otimes ((b_3 \leftharpoonup {a'_3}^2) \leftarrow h'_2) b'\end{aligned}$$

counit:

$$\epsilon : A \otimes H \otimes B \to K$$

$$a \otimes h \otimes b \mapsto \epsilon_A(a) \epsilon_H(h) \epsilon_B(b)$$

unit $1 \otimes 1 \otimes 1$ and antipode:

$$S : A \otimes H \otimes B \to A \otimes H \otimes B$$

$$a \otimes h \otimes b \mapsto (1 \otimes 1 \otimes S_B(b^1))(1 \otimes S_H(a^1 h b^2) \otimes 1)(S_A(a^2) \otimes 1 \otimes 1)$$

where $\rightharpoonup$, $\leftharpoonup$ are the coadjoint actions

$$b \rightharpoonup a = \langle (S_H({a_2}^1) \to a_1) S_A(a_3), b \rangle_B {a_2}^2$$

$$b \leftharpoonup a = \langle a, S_B(b_1)(b_3 \leftarrow S_H({b_2}^2)) \rangle_A {b_2}^1$$

and $\sharp$ is defined as:

$$b \sharp a := \langle a_1, {b_1}^1 \rangle_B {b_1}^2 {a_2}^1 \langle {a_2}^2, b_2 \rangle_A$$

We shall need the following Lemma.

Lemma

1. Suppose that Λ_A and Γ_A are a left and a right integral of A. Then

   $$B \to K, b \mapsto \langle \Lambda_A, b \rangle_B$$

   is a left integral and

   $$B \to K, b \mapsto \langle \Gamma_A, b \rangle_B$$

   is a right integral of B^*

2. Suppose that Λ_B and Γ_B are a left and a right integral of B. Then

   $$A \to K, a \mapsto \langle a, \Lambda_B \rangle_A$$

   is a left integral and

   $$A \to K, a \mapsto \langle a, \Gamma_B \rangle_A$$

   is a right integral of A^*.

3. If a_A^L (resp. a_A^R) is the left (resp. right) modular element of A and α_B^L (resp. α_B^R) is the left (resp. right) modular function of B, we have:
$$\langle a_A^L, b\rangle_A = \alpha_B^L(b) \qquad \langle a_A^R, b\rangle_A = \alpha_B^R(b)$$
4. If a_B^L (resp. a_B^R) is the left (resp. right) modular element of B and α_A^L (resp. α_A^R) is the left (resp. right) modular function of A, we have:
$$\langle a, a_A^L\rangle_B = \alpha_A^L(a) \qquad \langle a, a_B^R\rangle_B = \alpha_A^R(a)$$

Proof. We have by [32], Proposition 4.2:
$$\langle a, b_1\rangle_B\langle \Lambda_A, b_2\rangle_B = \langle a\Lambda_A, b\rangle_B = \epsilon_A(a)\langle \Lambda_A, b\rangle_B = \langle a, 1\rangle_B\langle \Lambda_A, b\rangle_B$$
We conclude: $b_1\langle \Lambda_A, b_2\rangle_B = 1\langle \Lambda_A, b\rangle_B$ The other verifications are similar. □

Theorem Suppose that Λ_A and Λ_H are left integrals of A resp. H and that Γ_B is a right integral of B. Then
$$\Lambda_A \otimes \Lambda_{H1}\iota_A(\Lambda_{H2}) \otimes \Gamma_B$$
is a left and a right integral of $A \otimes H \otimes B$, where ι_A is the integral character of A. In particular, $A \otimes H \otimes B$ is unimodular.

Proof. We can assume that the integrals involved are nonzero. Select a right integral ρ_A of A^* satisfying $\rho_A(\Lambda_A) = 1$ and define:
$$\omega_B : B \to K, b \mapsto \rho_A(S_B(b^1) \rightharpoonup \Lambda_A)\iota_A^{-1}(b^2)$$
Using the integral group element g_A of A, we have:
$$\begin{aligned}
\omega_B(b) &= \langle (S_H({\Lambda_{A2}}^1) \rightharpoonup \Lambda_{A1})S_A(\Lambda_{A3}), S_B(b^1)\rangle_B \rho_A({\Lambda_{A2}}^2)\iota_A^{-1}(b^2) \\
&= \langle (S_H(g_A) \rightharpoonup \Lambda_{A1})S_A(\Lambda_{A3}), S_B(b^1)\rangle_B \rho_A(\Lambda_{A2})\iota_A^{-1}(b^2) \\
&= \langle (S_H(g_A) \rightharpoonup a_A^R), S_B(b^1)\rangle_B \rho_A(\Lambda_A)\iota_A^{-1}(b^2) \\
&= \langle a_A^R, S_B(b^1)\rangle_B \iota_A^{-1}(b^2) \\
&= \langle a_A^L, b^1\rangle_B \iota_A^{-1}(b^2) \\
&= \langle a_A^R, b^1\rangle_A \iota_A^{-1}(b^2) = \alpha_B^R(b^1)\iota_A^{-1}(b^2) = \alpha_B^R(b)
\end{aligned}$$
We therefore see that ω_B is in this case the right modular function of B. Since left integrals corresponding to this character are precisely the right integrals, we see by Theorem 5.2 that the above element is a left integral. By applying part (2) of Theorem 5.2, we derive by a similar calculation that the element $\Lambda_A \otimes \iota_A(\Gamma_{H1})\Gamma_{H2} \otimes \Gamma_B$ is a right integral of $A \otimes H \otimes B$, where Γ_H is a right integral of H. Since H is commutative and cocommutative, H is unimodular, and left and right integrals of $A \otimes H \otimes B$ coincide. □

5.5 The above Theorem generalizes a result of D. Radford on the integrals of the Drinfel'd double construction (cf. [24], Thm. 4, p. 303). We note that the same result was proved by M. A. Hennings (cf. [7]) and K. Hoffmann (unpublished). Recall from [32], subsection 4.18 how the Drinfel'd double construction arises as a special case of the second construction: First, we see from Theorem 5.4 above that if we set $H = K$ in the second construction, then $\Lambda_A \otimes \Gamma_B$ is an integral of $A \otimes B$, where $\Lambda_A \in A$ is a left integral and $\Gamma_B \in B$ is a right integral. Since $A \otimes B$ is unimodular, this remains an integral if we pass to $(A \otimes B)^{op\ cop}$. Following [32]), we now have to interchange the tensorands A and B to obtain the Drinfel'd double of $A^{op\ cop}$. Therefore, $\Gamma_B \otimes \Lambda_A$ is a two-sided integral of the Drinfel'd double of $A^{op\ cop}$, which implies that an integral of the Drinfel'd double of A is of the form $\Lambda_B \otimes \Gamma_A$ for a left integral $\Lambda_B \in B$ and a right integral $\Gamma_A \in A$.

6 The Frobenius-Lusztig kernel of $U_q(sl(2))$

6.1 To illustrate the techniques developed in the previous section, we calculate in this section the integral for a finite-dimensional quotient of the deformed enveloping algebra $U_q(sl(2))$. Suppose that q is a primitive m-th root of unity in the field K. We assume that m is larger than 2. Note that the existence of a primitive root of unity implies that the characteristic of the field does not divide its order m. We introduce the number k by:

$$k = \begin{cases} m & \text{if } m \text{ is odd} \\ \frac{m}{2} & \text{if } m \text{ is even} \end{cases}$$

Let G be the cyclic group of order k generated by g. We set $H = K[G]$, the group ring of G, with its usual Hopf algebra structure. We shall turn the polynomial algebra $K[\theta]$ over the indeterminate θ into a left Yetter-Drinfel'd module over H via the module operation

$$g \rightharpoonup \theta^n = q^{2n}\theta^n$$

and the comodule cooperation

$$\delta_{K[\theta]}(\theta^n) = g^n \otimes \theta^n$$

It can be verified directly that $K[\theta]$ becomes an algebra in the category of left Yetter-Drinfel'd modules in this way.

Consider the tensor product algebra $K[\theta] \otimes K[\theta]$ formed inside the category of Yetter-Drinfel'd modules. We introduce a coalgebra structure on $K[\theta]$ by requiring $\Delta_{K[\theta]}$ to be the unique algebra homomorphism satisfying

$$\Delta_{K[\theta]}(\theta) = \theta \otimes 1 + 1 \otimes \theta$$

We define the counit $\epsilon_{K[\theta]} : K[\theta] \to K$ to be the unique algebra homomorphism annihilating θ, and define the antipode $S_{K[\theta]} : K[\theta] \to K[\theta]^{opp}$ to be the unique algebra homomorphism satisfying $S_{K[\theta]}(\theta) = -\theta$. Here $K[\theta]^{opp}$ means the opposite algebra in the categorical sense: The multiplication $\mu^{opp}_{K[\theta]}$ of $K[\theta]^{opp}$ is given by:

$$\mu^{opp}_{K[\theta]} = \mu_{K[\theta]} \circ \sigma_{K[\theta],K[\theta]}$$

where σ is the braiding as in subsection 2.2, and the unit of $K[\theta]^{opp}$ is not changed. Note that this opposite multiplication is different from the one used in subsection 4.5, where we used the inverse braiding in contrast to the ordinary braiding used here. These morphisms are H-linear and colinear, and one can verify directly that $K[\theta]$ is a Yetter-Drinfel'd Hopf algebra.

6.2 Consider the principal ideal (θ^k) generated by θ^k. It is immediate that this ideal is an H-submodule and a subcomodule. We repeat briefly why it is also a coideal. It follows from the Gaussian binomial formula (cf. [13], subsection 1.3.5, p. 10) that

$$\Delta_{K[\theta]}(\theta^n) = \sum_{i=0}^{n} q^{i(n-i)} \begin{bmatrix} n \\ i \end{bmatrix} \theta^i \otimes \theta^{n-i}$$

Here, we have used the so called Gaussian binomial coefficients that are defined recursively as:

$$\begin{bmatrix} n \\ 0 \end{bmatrix} = \begin{bmatrix} n \\ n \end{bmatrix} = 1 \qquad \begin{bmatrix} n+1 \\ i \end{bmatrix} = q^{-n} \begin{bmatrix} n \\ i \end{bmatrix} + q^{n-i+1} \begin{bmatrix} n \\ i-1 \end{bmatrix}$$

Since it can be shown (cf. [13], Lemma 34.1.2, p. 265) that the Gaussian binomial coefficients $\left[\begin{smallmatrix} k \\ i \end{smallmatrix}\right]$ vanish for $i = 1, \dots, k-1$, we have $\Delta_{K[\theta]}(\theta^k) = \theta^k \otimes 1 + 1 \otimes \theta^k$, which proves that (θ^k) is a coideal. Since it follows from the form of the antipode in the general case (cf. [32], Prop. 5.1, p. 56) or from a simple induction that we have

$$S_{K[\theta]}(\theta^n) = (-1)^n q^{n(n-1)} \theta^n,$$

the ideal (θ^k) is also invariant with respect to the antipode.

The factor algebra $A = K[\theta]/(\theta^k)$ is therefore again a Yetter-Drinfel'd Hopf algebra over H via the induced structures. It obviously has dimension k over K. We denote the dual basis of $1, \theta, \theta^2, \dots, \theta^{k-1}$ by $p_0, p_1, \dots, p_{k-1}$, thereby using the notation θ also for the equivalence class of θ in A.

Denote by B the dual of A modified as described in subsection 5.4. We can now carry out the second construction and obtain a Hopf algebra structure on $A \otimes H \otimes B$. We introduce the following elements:

$$E = \theta \otimes 1 \otimes 1, \quad K = 1 \otimes g \otimes 1, \quad F = 1 \otimes 1 \otimes \frac{p_1}{q^{-1} - q}$$

It is obvious that K is invertible with inverse $1 \otimes g^{-1} \otimes 1$. The following Proposition describes precisely how $A \otimes H \otimes B$ arises as a quotient of $U_q(sl(2))$:

Proposition The elements E, F, K and K^{-1} satisfy:

$$KK^{-1} = 1 = K^{-1}K$$

$$KE = q^2 EK \qquad KF = q^{-2} FK$$

$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}}$$

$$K^k = 1 \qquad E^k = 0 \qquad F^k = 0$$

and this constitutes a presentation of $A \otimes H \otimes B$ in terms of generators and relations.

Proof. (1) We show first that the elements E, F, K and K^{-1} in fact satisfy the above relations. Observe that p_1 is primitive. The relations are all rather obvious except for the fourth one. We have:

$$p_1 \rightharpoonup \theta = \langle \theta, p_1 \rangle_B 1 + \langle -\theta, p_1 \rangle_B 1 = 0$$

$$p_1 \leftharpoonup \theta = \langle \theta, -p_1 \rangle_A 1 + \langle \theta, p_1 \rangle_A 1 = 0$$

$$p_1 \sharp \theta = \langle \theta, p_1 \rangle_B g^{-1} + g \langle \theta, p_1 \rangle_A = g - g^{-1}$$

We therefore can calculate:

$$\begin{aligned} FE &= (1 \otimes 1 \otimes \frac{p_1}{q^{-1} - q})(\theta \otimes 1 \otimes 1) \\ &= \theta \otimes 1 \otimes \frac{p_1}{q^{-1} - q} + 1 \otimes (\frac{p_1}{q^{-1} - q} \sharp \theta) \otimes 1 \\ &= EF + \frac{K - K^{-1}}{q^{-1} - q} \end{aligned}$$

(2) Now suppose that X is the algebra generated by symbols E', F', K' and K'^{-1} subject to the above relations. It is clear that we have an algebra morphism $\pi : X \rightarrow A \otimes H \otimes B$. It is obvious from the relations that the elements $E'^i K'^n F'^j$ for $0 \leq i, j, n \leq k - 1$ generate X as a vector space. On the other hand, the elements $E^i K^n F^j$ for $0 \leq i, j, n \leq k - 1$ form a basis of $A \otimes H \otimes B$ by construction. Since a nontrivial linear relation for the primed system would map to a nontrivial linear relation for the unprimed system, both must form a basis. Therefore π is an isomorphism. □

6.3 In order to calculate the integral of $A \otimes H \otimes B$, we must find the integrals of the components first.

Lemma

1. θ^{k-1} is a nonzero left and right integral of A. The integral group element of A is g^{k-1}. The integral character of A is given by

$$\iota_A(g^n) = q^{2n(k-1)}$$

2. p_{k-1} is a nonzero left and right integral of B. The integral group element of B is $g^{-(k-1)}$. The integral character ι_B of B is equal to ι_A.

3. The sum $\sum_{i=1}^{k} g^i$ is a left and right integral of H.

Proof. The third assertion is obvious and extremely well known (cf. [19], Example 2.1.2, p. 17). The first assertion is also obvious. It remains to show the second assertion. As we have already pointed out above, we have:

$$\Delta_A(\theta^n) = \sum_{i=0}^{n} q^{i(n-i)} \begin{bmatrix} n \\ i \end{bmatrix} \theta^i \otimes \theta^{n-i}$$

The product of two elements of the dual basis is therefore equal to:

$$p_i p_j = q^{ij} \begin{bmatrix} i+j \\ i \end{bmatrix} p_{i+j}$$

In addition, we have by definition:

$$\delta_B(p_n) = p_n \otimes g^{-n} \qquad p_n \leftharpoonup g = q^{2n} p_n$$

The second assertion is now clear. $\square$

By Theorem 5.4, we now can obtain immediately the integral of our quotient:

Proposition The element

$$\sum_{i=1}^{k} q^{2i(k-1)} E^{k-1} K^i F^{k-1}$$

is a nonzero left and right integral of $A \otimes H \otimes B$.

This can of course also be verified directly.

References

[1] N. Andruskiewitsch: Notes on extensions of Hopf algebras, Canad. J. Math. 48 (1996), 3-42 (with an appendix by N. Andruskiewitsch and H.-J. Schneider)

[2] N. Andruskiewitsch, H.-J. Schneider: Hopf algebras of order p^2 and braided Hopf algebras of order p, preprint, 1996

[3] V. G. Drinfel'd: Quantum groups, in: Proceedings of the International Congress of Mathematicians, Volume I, 798-820, Amer. Math. Soc., Berkeley, 1987

[4] Y. Doi: Hopf modules in Yetter-Drinfel'd categories, preprint, 1997

[5] D. Fischman, S. Montgomery, H.-J. Schneider: Frobenius extensions of subalgebras of Hopf algebras, preprint, 1995, to appear in: Trans. Amer. Math. Soc.

[6] P. J. Freyd, D. N. Yetter: Braided compact closed categories with applications to low dimensional topology, Adv. Math. 77 (1989), 156-182

[7] M. A. Hennings: Invariants of links and 3-manifolds obtained from Hopf algebras, preprint, 1990

[8] J. C. Jantzen: Lectures on quantum groups, Grad. Stud. Math. 6, Amer. Math. Soc., Providence, R. I., USA, 1996

[9] A. Joyal, R. Street: Braided tensor categories, Adv. Math. 102 (1993), 20-78

[10] F. Kasch: Grundlagen einer Theorie der Frobeniuserweiterungen, Math. Ann. 127 (1954), 453-474

[11] M. Koppinen: Coideal subalgebras in Hopf algebras: Freeness, integrals, smash products, Comm. Algebra 21 (1993), 427-444

[12] R. G. Larson, M. E. Sweedler: An associative orthogonal bilinear form for Hopf algebras, Amer. J. Math. 91 (1969), 75-93

[13] G. Lusztig: Introduction to Quantum Groups, Progr. Math. 110, Birkhäuser, Basel, 1993

[14] G. Lusztig: Finite dimensional Hopf algebras arising from quantized universal enveloping algebras, J. Amer. Math. Soc. 3 (1990), 257-296

[15] G. Lusztig: Quantum groups at roots of 1, Geom. Dedicata 35 (1990), 89-114

[16] V. Lyubashenko: Tangles and Hopf algebras in braided categories, J. Pure Appl. Algebra 98 (1995), 245-278

[17] V. Lyubashenko: Modular transformations for tensor categories, J. Pure Appl. Algebra 98 (1995), 279-327

[18] S. Majid: Double bosonization of braided groups and the construction of $\mathbb{U}_q(g)$, preprint DAMTP/95-57, 1995

[19] S. Montgomery: Hopf Algebras and their Actions on Rings, CBMS Regional Conf. Ser. in Math. 82, Amer. Math. Soc., Providence, R. I., USA, 1993

[20] B. Pareigis: Einige Bemerkungen über Frobenius-Erweiterungen, Math. Ann. 153 (1964), 1-13

[21] B. Pareigis: Endliche Hopf-Algebren, Algebra-Ber., Uni-Druck, Munich, 1973

[22] D. Radford: The order of the antipode of a finite-dimensional Hopf algebra is finite, Amer. J. Math 98 (1976), 333-355

[23] D. Radford: The structure of Hopf algebras with a projection, J. Algebra 92 (1985), 322-347

[24] D. Radford: Minimal quasitriangular Hopf algebras, J. Algebra 157 (1993), 285-315

[25] D. Radford: Generalized double crossproducts associated with the quantized enveloping algebras, preprint, Chicago, 1991

[26] D. Radford: The trace function and Hopf algebras, J. Algebra 163 (1994), 583-622

[27] D. Radford, L. Kauffman: A necessary and sufficient condition for a finite-dimensional Drinfel'd double to be a ribbon Hopf algebra, J. Algebra 159 (1993), 98-114

[28] A. Rosenberg: Hopf algebras and Lie algebras in quasisymmetric categories, preprint, Moscow, 1978

[29] P. Schauenburg: Hopf modules and Yetter-Drinfel'd modules, J. Algebra 169 (1994), 874-890

[30] H.-J. Schneider: Lectures on Hopf algebras, Universidad de Cordoba Trabajos de Matematica, Ser. B, No. 31/95, Cordoba, Argentina, 1995

[31] Y. Sommerhäuser: Deformierte universelle Einhüllende, Diplomarbeit, Munich, 1994

[32] Y. Sommerhäuser: Deformed enveloping algebras, New York J. Math. 2 (1996), 35-58

[33] M. E. Sweedler: Hopf algebras, W. A. Benjamin, New York, 1969

[34] M. E. Sweedler: Integrals for Hopf algebras, Ann. of Math. (2) 89 (1969), 323-335

[35] M. Takeuchi: Matched pairs of groups and bismash products of Hopf algebras, Comm. Algebra 9 (1981), 841-882

[36] V. G. Turaev: Modular categories and 3-manifold invariants, Internat. J. Modern Phys. B 6 (1992), 1807-1824

[37] V. G. Turaev: Quantum invariants of knots and 3-manifolds, de Gruyter Stud. Math. 18, de Gruyter, Berlin, 1994

[38] D. N. Yetter: Quantum groups and representations of monoidal categories, Math. Proc. Cambridge Philos. Soc. 108 (1990), 261-290

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