University of South Alabama
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MA 540: Sheet 9

Differential Geometry

1. Let $f : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a regular surface element with principal curvatures $\kappa_1(u,v)$ and $\kappa_2(u,v)$, so that the Gaussian curvature is given by $K(u,v) = \kappa_1(u,v)\kappa_2(u,v)$. With a matrix $A \in SO(3)$ and a vector $c \in \mathbb{R}^3$, define a new surface element

$$\tilde{f}(u,v) := Af(u,v) + c$$

whose corresponding principal curvatures are denoted by $\tilde{\kappa}_1$ and $\tilde{\kappa}_2$ and whose Gaussian curvature is denoted by $\tilde{K}$. Show that

$$\tilde{\kappa}_1(u,v) = \kappa_1(u,v) \qquad \tilde{\kappa}_2(u,v) = \kappa_2(u,v) \qquad \tilde{K}(u,v) = K(u,v)$$

2. Let A be a symmetric 2×2-matrix with real entries. Consider the function

$$f(x,y) = (x,y)A\begin{pmatrix} x \\ y \end{pmatrix}$$

and the function $g(x,y) = x^2 + y^2$. The unit circle

$$K = \{(x,y) \mid g(x,y) = 1\}$$

is a compact set, so that the continuous function f attains its maximum and its minimum on K. Let $(x_1,y_1) \in K$ be a point where the maximum is attained, and $(x_2,y_2) \in K$ be a point where the minimum is attained. Show that these two points are eigenvectors for A. Show that the corresponding eigenvalues are real.

(Notes: Use Lagrange multipliers. Distinguish the two cases in which f is constant on K and in which f is not constant on K. Recall that in class, we used the converse of this result and showed that the eigenvalues are the maximum and the minimum of f.)

3. By definition, an ellipsoid consists of the points (x,y,z) that satisfy

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

where a, b, and c are strictly positive constants.

   (a) Show that the ellipsoid can be parametrized by

$$f(u,v) = (a\cos(u)\sin(v), b\sin(u)\sin(v), c\cos(v))$$

   where $u \in [0, 2\pi)$ and $v \in [0, \pi]$.

(b) Show that for this parametrization, the unit normal vector is given by $\nu = \mu/\|\mu\|$, where

$$\mu := \frac{\partial f}{\partial u} \times \frac{\partial f}{\partial v} = -\sin(v)(bc\cos(u)\sin(v), ac\sin(u)\sin(v), ab\cos(v))$$

(c) Show that the first fundamental form is given by

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} =$$
$$\begin{pmatrix} a^2\sin^2(u)\sin^2(v) + b^2\cos^2(u)\sin^2(v) & (b^2-a^2)\sin(u)\sin(v)\cos(u)\cos(v) \\ (b^2-a^2)\sin(u)\sin(v)\cos(u)\cos(v) & a^2\cos^2(u)\cos^2(v) + b^2\sin^2(u)\cos^2(v) + c^2\sin^2(v) \end{pmatrix}$$

(d) Show that the second fundamental form is given by

$$\begin{pmatrix} L & M \\ M & N \end{pmatrix} = \frac{1}{\|\mu\|}\begin{pmatrix} abc\sin^3(v) & 0 \\ 0 & abc\sin(v) \end{pmatrix}$$

(e) Show that the Gaussian curvature is given by the formula

$$K = \frac{a^2b^2c^2}{(b^2c^2\cos^2(u)\sin^2(v) + a^2c^2\sin^2(u)\sin^2(v) + a^2b^2\cos^2(v))^2}$$

Conclude that the Gaussian curvature is positive.

(f) In the case where $a = b = c$, the ellipsoid becomes a sphere. Show that in this case, the Weingarten map is $L = \frac{1}{a}\,\mathrm{id}$. From this, compute the principal curvatures and the Gaussian curvature.

Due date: Wednesday, November 9, 2011. Please write your solution on letter-sized paper, and write your name on your solution. It is not necessary to submit this sheet with your solution.