University of South Alabama **Fall Semester 2011**
Y. Sommerhäuser **MA 540: Sheet 8**

Differential Geometry

1. We have seen in class that the torus can be parametrized in the form

$$f(u,v) = ((a + b\cos(u))\cos(v), (a + b\cos(u))\sin(v), b\sin(u))$$

where $0 < b < a$.

(a) Find the Gram matrix of the first fundamental form (which is also called the metric tensor or the measure tensor)

$$G(u,v) = \begin{pmatrix} \langle \frac{\partial f}{\partial u}(u,v), \frac{\partial f}{\partial u}(u,v)\rangle & \langle \frac{\partial f}{\partial u}(u,v), \frac{\partial f}{\partial v}(u,v)\rangle \\ \langle \frac{\partial f}{\partial v}(u,v), \frac{\partial f}{\partial u}(u,v)\rangle & \langle \frac{\partial f}{\partial v}(u,v), \frac{\partial f}{\partial v}(u,v)\rangle \end{pmatrix}$$

for the torus.

(b) The xy-plane intersects the torus in two circles. Use the formula

$$L = \int_a^b \sqrt{\dot{\gamma}(t)^T G(\gamma(t))\dot{\gamma}(t)}dt$$

for the length of a curve γ on the surface (i.e., a curve $(u,v) = \gamma(t)$ in parameter space) to find the length of the larger one of these circles. Explain why the result agrees with the standard formula for the circumference of a circle. (Hint: This formula is discussed on page 60 of the textbook.)

2. For the parametrization of the torus given in the previous problem, find the matrix representation of the Weingarten map L with respect to the basis $\frac{\partial f}{\partial u}(u,v)$ and $\frac{\partial f}{\partial v}(u,v)$ of the tangent space. (Hint: This is a 2×2-matrix. The Weingarten map is defined on page 68 of the textbook.)

3. Suppose that

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is a 2×2-matrix.

(a) Show that the characteristic polynomial $\chi(t)$ of A is given by the formula

$$\chi(t) = t^2 - \mathrm{tr}(A)t + \det(A) = t^2 - (a + d)t + (ad - bc)$$

(b) Prove the Cayley-Hamilton theorem for 2×2-matrices, which asserts that

$$\chi(A) = A^2 - (a+d)A + (ad - bc) = 0$$

(Notes: The Cayley-Hamilton theorem holds more generally for $n \times n$-matrices, but is more difficult to prove in this setting.)

Due date: Monday, October 31, 2011. Please write your solution on letter-sized paper, and write your name on your solution. It is not necessary to submit this sheet with your solution.