University of South Alabama **Fall Semester 2011**
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Differential Geometry

1. Recall that a surface element $f : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ is required to be regular in the sense that it is continuously differentiable and its Jacobi matrix has rank 2 everywhere in U. A surface element $\tilde{f} : \tilde{U} \subset \mathbb{R}^2 \to \mathbb{R}^3$ is called a reparametrization of f if there is a diffeomorphism $\varphi : \tilde{U} \to U$, i.e., a bijection which is continuously differentiable with a continuously differentiable inverse, such that we have
$$\tilde{f}(u) = f(\varphi(u))$$
for all $u \in \tilde{U}$. Show that 'reparametrization' defines an equivalence relation on the set of surface elements. For this, you have by definition to show the following three things:
   (a) Reflexivity: Every surface element is a reparametrization of itself.
   (b) Symmetry: If $\tilde{f}$ is a reparametrization of f, then f is a reparametrization of $\tilde{f}$.
   (c) Transitivity: If $\tilde{f}$ is a reparametrization of f and $\bar{f}$ is a reparametrization of $\tilde{f}$, then $\bar{f}$ is a reparametrization of f. (6 points)

2. Recall that for a surface element $f : U \subset \mathbb{R}^2 \to \mathbb{R}^3$, the surface area is given by the formula
$$A = \int\int_U \|\frac{\partial f}{\partial u}(u,v) \times \frac{\partial f}{\partial v}(u,v)\| dudv$$
(cf. J. Rogawski, Calculus, Sec. 16.4, Thm. 1, p. 982).
   (a) Show that the surface area can also be written in the form
$$A = \int\int_U \sqrt{\det(G(u,v))} dudv$$
where
$$G(u,v) = \begin{pmatrix} \langle \frac{\partial f}{\partial u}(u,v), \frac{\partial f}{\partial u}(u,v)\rangle & \langle \frac{\partial f}{\partial u}(u,v), \frac{\partial f}{\partial v}(u,v)\rangle \\ \langle \frac{\partial f}{\partial v}(u,v), \frac{\partial f}{\partial u}(u,v)\rangle & \langle \frac{\partial f}{\partial v}(u,v), \frac{\partial f}{\partial v}(u,v)\rangle \end{pmatrix}$$
is the Gram matrix of the first fundamental form (which is also called the metric tensor or the measure tensor). (Hint: J. Rogawski, Calculus, Sec. 12.4, Eq. 10, p. 707, see also J. Stewart, Calculus, Sec. 12.4, Exerc. 44, p. 821, which we once did in class.)

(b) Show that if $\tilde{f}$ is a reparametrization of f as in the first problem, then $\tilde{f}$ has the same area as f. (Hint: W. Kühnel, Differential Geometry, Sec. 3A, Lem. 3.3, p. 61.) (8 points)

3. If $S^1 \subset \mathbb{R}^2$ denotes the unit circle, then the (Clifford) torus is the set

$$T_C := S^1 \times S^1 = \{(\alpha, \beta, \rho, \sigma) \mid \alpha^2 + \beta^2 = \rho^2 + \sigma^2 = 1\} \subset \mathbb{R}^4$$

On the other hand, if $0 < b < a$, the corresponding torus of revolution is

$$T_R = \{(x, y, z) \mid (a^2 - b^2 + x^2 + y^2 + z^2)^2 = 4a^2(x^2 + y^2)\} \subset \mathbb{R}^3$$

Show that

$$\tau : T_C \to T_R, \ (\alpha, \beta, \rho, \sigma) \mapsto ((a + b\alpha)\rho, (a + b\alpha)\sigma, b\beta)$$

is a bijection. (6 points)

Due date: Monday, October 24, 2011. Please write your solution on letter-sized paper, and write your name on your solution. It is not necessary to submit this sheet with your solution.