University of South Alabama **Fall Semester 2011**
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Differential Geometry

1. For an $n \times n$-matrix $A = (a_{ij})_{i,j=1,\ldots,n}$, we define the row-sum norm (also called the maximum absolute row-sum norm) as the number

$$\|A\| := \max \left\{ \sum_{j=1}^{n} |a_{ij}| \middle| i = 1, \ldots, n \right\}$$

   (a) Show that the row-sum norm really is a norm.
   (Notes: By definition, a norm needs to satisfy the following three properties: 1. $\|A + B\| \leq \|A\| + \|B\|$, 2. $\|\lambda B\| = |\lambda| \|B\|$ for a real number λ, 3. $\|A\| = 0$ if and only if $A = 0$.)

   (b) Show that for another $n \times n$-matrix B, we have $\|AB\| \leq \|A\| \|B\|$.

   (c) Suppose that $\sum_{k=0}^{\infty} a_k x^k$ is a power series with convergence radius r. Use Cauchy's criterion to show that if $\|A\| < r$, then

$$\sum_{k=0}^{\infty} a_k A^k$$

   converges.
   (Notes: This is a first crude example of so-called functional calculus, the question of when a matrix, or more generally a linear operator, can be plugged into a function, here our power series. This question of course arises very often, especially in quantum mechanics.)

   (d) Show that

$$\sum_{k=0}^{\infty} \frac{1}{k!} \begin{pmatrix} 0 & \alpha \\ -\alpha & 0 \end{pmatrix}^k = \begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{pmatrix}$$

   (Notes: First, decide what A is. Second, compute A^2. Third, figure out what an arbitrary even and an arbitrary odd power of A looks like. Finally, break the series into even and odd terms and use the standard power series for sine and cosine, as in Rogawski, Sec. 10.7, Example 2, p. 607.) (6 points)

2. A linear map $f : \mathbb{R}^n \to \mathbb{R}^n$ is called skew-symmetric, or briefly skew, if

$$\langle f(x), y \rangle = -\langle x, f(y) \rangle$$

   for all $x, y \in \mathbb{R}^n$.

(a) Show that $\det(f) = (-1)^n \det(f)$. Conclude that f cannot be bijective if n is odd.

(b) Show that f cannot have nonzero real eigenvalues.

(c) Show that the image $f(\mathbb{R}^n)$ is the orthogonal complement of the kernel $\ker(f)$: $f(\mathbb{R}^n) = \ker(f)^\perp$

(d) Show that the restriction of f to $f(\mathbb{R}^n)$ induces a bijection from $f(\mathbb{R}^n)$ to itself.

(e) Show that the rank of f is even.

(Hints: Such an f has the form $f(x) = Ax$ for a skew-symmetric $n \times n$-matrix A. This description is completely equivalent. Use the dimension formula in the third part.) (6 points)

3. Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is skew-symmetric, and set $g := f \circ f$.

(a) Show that g is symmetric in the sense that $\langle g(x), y\rangle = \langle x, g(y)\rangle$.

(b) It is an important theorem about symmetric linear maps that their eigenvalues are real and that they are diagonalizable, i.e., the whole space is the direct sum of the eigenspaces. Show that in the present case, all nonzero eigenvalues are strictly negative.

(c) Show that $\ker(f) = \ker(g)$ and $f(\mathbb{R}^n) = g(\mathbb{R}^n)$.

(d) Show that every eigenspace of g is invariant under f.

(e) Suppose that λ is a nonzero eigenvalue of g with a corresponding eigenvector v that is normalized so that it has length 1. Show that v and $f(v)$ are linearly independent and span a subspace that is invariant under f. Show that $v, \frac{1}{\mu} f(v)$ is an orthonormal basis of this subspace, where $\mu = \sqrt{-\lambda}$.

(f) Show that the matrix representation of the restriction of f to this subspace with respect to the basis $v, \frac{1}{\mu} f(v)$ is

$$\begin{pmatrix} 0 & -\mu \\ \mu & 0 \end{pmatrix}$$

(g) Show that there is an orthonormal basis of $\mathbb{R}^n$ for which the matrix representation of f is block-diagonal with blocks of the form

$$\begin{pmatrix} 0 & -\mu \\ \mu & 0 \end{pmatrix}$$

(Hint: Consider the restriction of f to the eigenspaces of g.)(8 points)

Due date: Wednesday, October 12, 2011. Please write your solution on letter-sized paper, and write your name on your solution. It is not necessary to submit this sheet with your solution.