**University of South Alabama**
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**Fall Semester 2011**
**MA 540: Sheet 5**

# Differential Geometry

1. Suppose that $c(t) = (f(t), g(t))$ is a regular plane curve, defined on the compact interval $[a, b]$, that satisfies $f(t)^2 + g(t)^2 = r^2$ for some constant $r > 0$, i.e., is contained in a circle. Suppose that for some parameter $t_0$, we have $c(t_0) = (r\cos(\varphi_0), r\sin(\varphi_0))$.

   (a) Show that there exists a continuously differentiable function $\varphi : [a, b] \to \mathbb{R}$ such that
   $$c(t) = (r\cos(\varphi(t)), r\sin(\varphi(t)))$$
   for all $t \in [a, b]$, and $\varphi(t_0) = \varphi_0$.

   (b) Show that such a function is unique.

   Notes: This result is called the main theorem on circular motion. The point here is that the function needs to be continuously differentiable. This problem can be approached using inverse trigonometric functions. The difficulty with this approach is that the inverse trigonometric functions are only inverses of certain restrictions of the trigonometric functions. Therefore, the recommended approach is to define
   $$\varphi(t) = \varphi_0 + \frac{1}{r^2}\int_{t_0}^{t} f(\tau)\dot{g}(\tau) - \dot{f}(\tau)g(\tau)d\tau$$
   and to show that the two functions $f(t)\cos(\varphi(t)) + g(t)\sin(\varphi(t))$ and $f(t)\sin(\varphi(t)) - g(t)\cos(\varphi(t))$ are constant. Note that this result was implicitly used, for $\dot{c}$ instead of $c$, when we reconstructed a plane curve from its curvature, as described on page 15 and 16 of the textbook. (6 points)

2. Show that a Frenet space curve with constant curvature and constant torsion is a helix.

   Hints: Use Theorem 2.11 in the textbook, which is known as Lancret's theorem and which we proved in class. By rotating the coordinate system, you can assume that the vector that appears there is the unit vector in the $z$-direction. Then apply the previous problem. (8 points)

3. The curve
   $$c(t) = (e^t\cos(t), e^t\sin(t), e^t)$$
   is called a conical helix.

(a) Compute the curvature $\kappa(t)$ and the torsion $\tau(t)$.

(b) Verify Lancret's theorem (Theorem 2.11) by showing that $\tau(t)/\kappa(t)$ is constant and exhibiting a vector that makes a constant angle with the tangent vector of the conical helix.

(c) Sketch the curve and its projection to the $(x, y)$-plane. (6 points)

Due date: Wednesday, October 5, 2011. Please write your solution on letter-sized paper, and write your name on your solution. It is not necessary to submit this sheet with your solution.