University of South Alabama **Fall Semester 2011**
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Differential Geometry

1. Let $\lambda : [0, L] \to \mathbb{R}$ be a continuous function. For $s \in [0, L]$, define

$$c(s) := \left(\int_0^s \cos \left(\int_0^r \lambda(t)dt \right) dr, \int_0^s \sin \left(\int_0^r \lambda(t)dt \right) dr \right)$$

so that we have defined a curve c on the interval $[0, L]$. Show that this curve has the following properties:

   (a) c is twice continuously differentiable. (1 point)

   (b) c is parametrized by arc-length. (1 point)

   (c) $c(0) = (0, 0)$ and $c'(0) = (1, 0)$, so that the given coordinate system is an adapted coordinate system for our curve at $s = 0$. (1 point)

   (d) If $\kappa(s)$ is the curvature of $c(s)$, then we have $\kappa(s) = \lambda(s)$; in other words, the curvature function is exactly the function that was given at the beginning. (3 points)

2. Suppose that $c(t)$ is a regular curve defined on the compact interval $[a, b]$. Another regular curve $d(t)$ that is also defined on the interval $[a, b]$ is called an involute of c if, for all $t \in [a, b]$, the tangent line to c at $c(t)$ passes through $d(t)$ at a right angle, in the sense that $\langle \dot{c}(t), \dot{d}(t) \rangle = 0$.

   (a) Show that we can construct an involute of c by defining

$$d(t) := c(t) - (s(t) - k)e_1(t)$$

   where k is an arbitrary constant, $e_1(t) = \dot{c}(t)/\|\dot{c}(t)\|$ is the unit tangent vector of c and

$$s(t) = \int_a^t \|\dot{c}(\tau)\| d\tau$$

   is the arc-length function of c.

   (b) Show that every involute $d(t)$ of $c(t)$ has the form presented in the first part for a suitable value of the constant k. (6 points)

Notes: Observe that for the specific involute for which $k = 0$, we have $\|d(t) - c(t)\| = s(t)$, as in the animations that we watched in the lecture.

3. Suppose that $c(t)$ is a regular curve defined on the compact interval $[a, b]$. Suppose that its curvature function $\kappa(t)$ never vanishes and that the derivative $\dot{\kappa}(t)$ is strictly positive on $[a, b]$. Consider the evolute

$$\gamma(t) = c(t) + \frac{1}{\kappa(t)} e_2(t)$$

and form an involute of this evolute as in the preceding problem, using the constant $k = 1/\kappa(a)$. Show that this involute is the original curve.

Notes: This fact is often summarized by saying that a curve is the involute of its evolute. Recall from the lecture that our assumption $\dot{\kappa}(t) > 0$ implies that the evolute $\gamma(t)$ is regular. (8 points)

Due date: Wednesday, September 14, 2011. Please write your solution on letter-sized paper, and write your name on your solution. It is not necessary to submit this sheet with your solution.