University of South Alabama **Fall Semester 2011**
Y. Sommerhäuser **MA 540: Sheet 2**

Differential Geometry

1. (a) The curve
$$l(t) = (a, b) + t(c, d)$$
describes a line. Show that this is an arc-length parametrization if and only if (c, d) is a unit vector. Furthermore, show that a line has curvature zero.

   (b) Conversely, suppose that $c(s)$ is parametrized by arc-length and that it has zero curvature. Under the assumption that it is defined for all values of s, show that
$$c(s) = c(0) + sc'(0)$$
for all s.

   Notes: One possibility to approach this problem is via the Frenet-Serret equations. (6 points)

2. (a) For a positive number $R > 0$, the parametric curve
$$c_1(\varphi) = (a, b) + R(\cos(-\varphi/R), \sin(-\varphi/R))$$
is a circle with center (a, b) and radius R. Here the domain of c_1 is the interval $[0, 2\pi R]$. Show that this parametrization is an arc-length parametrization.

   (b) Show that the curvature of this circle is at every point equal to $-1/R$. In particular, it is constant and negative.

   (c) If $c_2(s)$ is a second curve that is parametrized by arc-length, show that for every point $c_2(s_2)$ on this curve at which the curvature $\kappa(s_2)$ of c_2 is negative there is a unique circle, called the osculating circle, that has contact of order 2 with the curve. Show that the center of this circle is
$$c_2(s_2) + \frac{1}{\kappa(s_2)} e_2(s_2)$$
where e_2 is the second vector in the Frenet frame determined by c_2.

   Notes: Recall that we have seen this in the lecture in the case of positive curvature, where the parametrization of the circle differed by a minus sign. Recall also that two curves $c_1(s)$ and $c_2(s)$ that are both parametrized by arc-length are said to have contact of order k if
$$c_1(s_1) = c_2(s_2) \qquad c_1'(s_1) = c_2'(s_2) \qquad \ldots \qquad c_1^{(k)}(s_1) = c_2^{(k)}(s_2)$$

for two parameter values s_1 and s_2. Note that there is a minor difference to the definition in the textbook and in the lecture, where s_1 and s_2 were both assumed to be zero, something that can always be accomplished by having the curve 'start' at s_1 resp. s_2. To solve the last part of the problem, note that the equation $c_1'(\varphi_1) = c_2'(s_2)$ can be used to find φ_1, whereas the equation $c_1''(\varphi_1) = c_2''(s_2)$ can be used to find the radius, which needs to be positive. Finally, the equation $c_1(\varphi_1) = c_2(s_2)$ can be used to find the center. (6 points)

3. Recall from the first lecture that the tractrix is parametrized by

$$c(t) = (t - \tanh(t), \operatorname{sech}(t))$$

where we assume that $t \in (-\infty, \infty)$, so that we get both branches.

(a) Show that the curvature $\kappa(t)$ of the tractrix is $\kappa(t) = |\operatorname{csch}(t)|$, the absolute value of the hyperbolic cosecant.

(b) Show that the evolute of the tractrix is the catenary $(t, \cosh(t))$, i.e., the graph of the hyperbolic cosine. (8 points)

Notes: The evolute is the curve that is formed by the centers of the osculating circles. As we have seen in the preceding problem, at least in the case of negative curvature, it is given by the parametrization

$$\gamma(t) = c(t) + \frac{1}{\kappa(t)} e_2(t)$$

where $\kappa(t)$ is the curvature of $c(t)$ and e_2 is the second vector in the Frenet frame. Note that this is also the case if c is not parametrized by arc-length, because this formula involves only geometric properties. (Think about this.) Note also that in the lecture, we have treated only the right branch of the tractrix, and it requires a symmetry argument to see that this parametrization also covers the left branch. It is necessary to be careful about sign differences between the two branches. It is also necessary to use Exercise 1 to Chapter 2 in the textbook, which we did in class.

Due date: Wednesday, September 7, 2011. Please write your solution on letter-sized paper, and write your name on your solution. It is not necessary to submit this sheet with your solution.