University of South Alabama **Fall Semester 2011**
Y. Sommerhäuser **MA 540: Sheet 12**

Differential Geometry

1. We have seen in Problem 3 on Sheet 11 that the Gram matrix of the first fundamental form for the pseudosphere is

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} \frac{1}{w^2} & 0 \\ 0 & \frac{1}{w^2} \end{pmatrix}$$

for $w \in [1, \infty)$ and $\varphi \in [0, 2\pi)$. This parameter domain corresponds only to a part of the upper half-plane, which consists of the parameter domain $w \in (0, \infty)$ and $\varphi \in (-\infty, \infty)$. However, we can think of the upper half-plane abstractly and use the area formula from Problem 2 on Sheet 7 to compute area not only on the pseudosphere, but rather anywhere in the upper half-plane.

   (a) In this sense, find the area of the set

$$T := \{(w, \varphi) \mid -1 \leq \varphi \leq 1,\ 1 - \varphi^2 \leq w^2\}$$

   (Hint: Use trigonometric substitution, as explained in Section 7.4 of Rogawski's calculus textbook, to evaluate the integral.)

   (b) Sketch T as a subset of the upper half-plane, and explain in which sense T can be viewed as a triangle with a 'point at infinity'.

   (c) Determine the angles α, β, γ of this triangle, and explain why your result is consistent with the general fact that the area of a triangle in the upper half-plane is $\pi - \alpha - \beta - \gamma$. (8 points)

2. Consider the helicoid

$$f(u, v) = (v \cos(u), v \sin(u), \lambda u)$$

   (a) Show that the helicoid is a ruled surface and find its standard parameters $c(u)$ and $X(u)$. Prove that these are indeed standard parameters.

   (b) Compute the invariants F, λ, and J from Theorem 3.22 of the textbook for the case of the helicoid. (6 points)

3. Consider the hyperboloid

$$H := \{(x, y, z) \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1\}$$

where a, b, and c are positive constants. Show that the mapping

$$S^1 \times \mathbb{R} \to H,\ \ (\alpha, \beta, t) \mapsto (a\alpha - ta\beta, b\beta + tb\alpha, tc)$$

is a bijection, where

$$S^1 := \{(\alpha, \beta) \mid \alpha^2 + \beta^2 = 1\}$$

is the unit circle. (6 points)

Due date: Wednesday, December 7, 2011. Please write your solution on letter-sized paper, and write your name on your solution. It is not necessary to submit this sheet with your solution.