University of South Alabama **Fall Semester 2011**
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Differential Geometry

1. Consider the curve
$$c(t) = (r(t), h(t)) = (k \sinh(t), \int_0^t \sqrt{1 - k^2 \cosh^2(u)} du)$$
in the (x, z)-plane, where $0 < k < 1$.
   (a) Find the values of t for which the curve is defined. Find the points where the curve intersects the z-axis and the points on the curve that have the greatest distance from the z-axis. Then sketch the curve, explaining the features of the sketch.
   (b) Define $\kappa := \sqrt{1 - k^2} < 1$ and $\varphi(w) := \sinh^{-1}(\frac{\kappa}{k} \cos(w))$. Using the elliptic integrals from Problem 1 on Sheet 10, show that
$$r(\varphi(w)) = \kappa \cos(w)$$
$$h(\varphi(w)) = E(\kappa, w) - F(\kappa, w) + C$$
   where $C = F(\kappa, \frac{\pi}{2}) - E(\kappa, \frac{\pi}{2})$. Then sketch the reparametrized curve $c(\varphi(w))$, noting that this curve is now defined for all values of w.
   (c) Sketch the surfaces of revolution that arise when $c(t)$ resp. $c(\varphi(w))$ are rotated around the z-axis. Then compute their Gaussian curvatures. How are they related? (6 points)

2. Consider the curve
$$c(t) = (r(t), h(t)) = (k \cosh(t), \int_0^t \sqrt{1 - k^2 \sinh^2(u)} du)$$
in the (x, z)-plane, where $0 < k < 1$.
   (a) Find the values of t for which the curve is defined. Show that the curve does not intersect the z-axis. Find the points on the curve that have the greatest and the smallest distance from the z-axis. Then sketch the curve, explaining the features of the sketch.
   (b) Define $\kappa := \frac{1}{\sqrt{1+k^2}} < 1$ and $\varphi(w) := \sinh^{-1}(\frac{1}{k} \cos(w))$. Using the elliptic integrals from Problem 1 on Sheet 10, show that
$$r(\varphi(w)) = \frac{1}{\kappa} \sqrt{1 - \kappa^2 \sin^2(w)}$$
$$h(\varphi(w)) = \frac{1}{\kappa} E(\kappa, w) - \frac{1}{\kappa} F(\kappa, w) + C$$

where $C = \frac{1}{\kappa}F(\kappa, \frac{\pi}{2}) - \frac{1}{\kappa}E(\kappa, \frac{\pi}{2})$. Then sketch the reparametrized curve $c(\varphi(w))$, noting that this curve is now defined for all values of w.

(c) Sketch the surfaces of revolution that arise when $c(t)$ resp. $c(\varphi(w))$ are rotated around the z-axis. Then compute their Gaussian curvatures. How are they related? (8 points)

3. Recall that we have seen in Problem 3 on Sheet 1 that the tractrix can be parametrized as

$$c(t) = (e^t, \int_0^t \sqrt{1 - e^{2x}}dx)$$

for $t \in (-\infty, 0]$.

(a) Show that

$$\int_0^t \sqrt{1 - e^{2x}}dx = \sqrt{1 - e^{2t}} - \cosh^{-1}(e^{-t})$$

(b) Show that

$$d(w) = \left(\frac{1}{w}, \sqrt{1 - \frac{1}{w^2}} - \cosh^{-1}(w)\right)$$

for $w \in [1, \infty)$ is an (orientation-reversing) reparametrization of the tractrix.

(c) Conclude that

$$f(w, \varphi) = \left(\frac{1}{w}\cos(\varphi), \frac{1}{w}\sin(\varphi), \sqrt{1 - \frac{1}{w^2}} - \cosh^{-1}(w)\right)$$

for $w \in [1, \infty)$ and $\varphi \in [0, 2\pi)$ is a parametrization of the pseudosphere.

(d) Show that in this parametrization, the Gram matrix of the first fundamental form is

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} \frac{1}{w^2} & 0 \\ 0 & \frac{1}{w^2} \end{pmatrix}$$

(Notes: Use the formulas for surfaces of revolution given on page 79 of the textbook. This Gram matrix is also known as the metric tensor for the upper-half plane model of hyperbolic geometry.) (6 points)

Due date: Wednesday, November 30, 2011. Please write your solution on letter-sized paper, and write your name on your solution. It is not necessary to submit this sheet with your solution.