University of South Alabama **Fall Semester 2011**
Y. Sommerhäuser **MA 540: Sheet 10**

Differential Geometry

1. Find a parametrization of the ellipse
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$
in terms of trigonometric functions and use this parametrization to show that the circumference of the ellipse is given by
$$b \int_0^{2\pi} \sqrt{1 - k^2 \sin^2(t)} dt$$
where $k = \sqrt{1 - \frac{a^2}{b^2}}$.
(Notes: For this reason, the integral appearing above is called an elliptic integral; it cannot be solved in terms of elementary functions. In fact, it is called an elliptic integral of the second kind; the three kinds are given respectively as
$$\begin{aligned} F(k,t) &= \int_0^t \frac{1}{\sqrt{1 - k^2 \sin^2(\phi)}} d\phi \\ E(k,t) &= \int_0^t \sqrt{1 - k^2 \sin^2(\phi)} d\phi \\ \Pi(k,t) &= \int_0^t \frac{1}{(1 + n \sin^2(\phi))\sqrt{1 - k^2 \sin^2(\phi)}} d\phi \end{aligned}$$
These three kinds of elliptic integrals are called the Legendre normal forms.) (6 points)

2. Using the elliptic integrals introduced in the preceding problem, consider the curve
$$c(t) = (r(t), h(t)) = (k \cos(t), E(k,t))$$
in the (x, z)-plane for $k < 1$.
   (a) Find the points where the curve intersects the z-axis and the points on the curve that have the greatest distance from the z-axis. Then sketch the curve.

(b) Sketch the surface of rotation that arises when $c(t)$ is rotated around the z-axis. Then compute its Gaussian curvature. (6 points)

3. As in the preceding problem, consider the curve

$$c(t) = (r(t), h(t)) = (k\cos(t), E(k,t))$$

in the (x,z)-plane, but now for $k > 1$.

(a) Find the values of t for which the curve is defined. Show that the curve does not intersect the z-axis. Find the points on the curve that have the greatest and the smallest distance from the z-axis. Then sketch the curve.

(b) Define $\kappa := 1/k < 1$ and define $\varphi(w) := \arcsin(\kappa \sin(w))$. Show that

$$\begin{aligned} r(\varphi(w)) &= \frac{1}{\kappa}\sqrt{1-\kappa^2\sin^2(w)} \\ h(\varphi(w)) &= \frac{1}{\kappa}E(\kappa,w) + \frac{\kappa^2-1}{\kappa}F(\kappa,w) \end{aligned}$$

Then sketch the reparametrized curve $c(\varphi(w))$, noting that this curve is now defined for all values of w.

(c) Sketch the surfaces of rotation that arise when $c(t)$ resp. $c(\varphi(w))$ are rotated around the z-axis. Then compute their Gaussian curvatures. How are they related? (8 points)

Due date: Wednesday, November 16, 2011. Please write your solution on letter-sized paper, and write your name on your solution. It is not necessary to submit this sheet with your solution.