University of South Alabama **Fall Semester 2011**
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Differential Geometry

1. In this problem, all curves considered are defined on compact intervals of the form $[a, b]$. Recall that a curve is regular if it is continuously differentiable and its derivative never vanishes. A curve d defined on $[a', b']$ is called a reparametrization of a curve c defined on $[a, b]$ if there is a bijection $\varphi : [a', b'] \to [a, b]$ which is continuously differentiable with strictly positive derivative such that we have

$$d(t) = c(\varphi(t))$$

for all $t \in [a', b']$. Show that 'reparametrization' defines an equivalence relation on the set of regular curves. For this, you have by definition to show the following three things:

   (a) Reflexivity: Every curve is a reparametrization of itself.

   (b) Symmetry: If d is a reparametrization of c, then c is a reparametrization of d.

   (c) Transitivity: If d is a reparametrization of c and e is a reparametrization of d, then e is a reparametrization of c. (6 points)

2. Suppose that c is a regular curve defined on $[a, b]$. As we know from calculus, its arc length is

$$L = \int_a^b \|\dot{c}(t)\| dt$$

Show that if d is a reparametrization of c as in the first problem, then d has the same arc length as c. (6 points)

3. If one rotates the parametrization of the tractrix given in class clockwise by 90 degrees, so that the x-axis and the y-axis are interchanged, one obtains the curve

$$d(r) := (\operatorname{sech}(r), \tanh(r) - r)$$

where we assume that $r \in [0, \infty)$. Show that the curve

$$c(t) := (\exp(t), \int_0^t \sqrt{1 - \exp(2x)} dx)$$

defined on $(-\infty, 0]$ is a reparametrization of this curve. (8 points)

Notes: The definition of reparametrization given in the first problem also applies here, although here the intervals of definition are not compact. This problem connects the treatment of the tractrix given in class, which was based on Exercise 73 in Section 11.1 of Rogawski's calculus textbook, with the treatment of the tractrix given on pages 11 and 12 in our textbook. Note that there is a mistake in our textbook: The parametrization given by Kühnel is a parametrization of the lower branch, not the upper. Note also that the length ℓ introduced in class is equal to 1 in this problem.

Due date: Wednesday, August 31, 2011. Please write your solution on letter-sized paper, and write your name on your solution. It is not necessary to submit this sheet with your solution.