University at Buffalo **Spring Semester 2014**
Yorck Sommerhäuser **MTH 819: Sheet 8**

Hopf Algebras

Problem 1: Show that every separable algebra is a Frobenius algebra.(4 points)

Problem 2: Suppose that A is a finite-dimensional semisimple algebra with Wedderburn decomposition

$$A = \bigoplus_{i=1}^{k} I_i, \qquad I_i \cong M(n_i \times n_i, D_i)$$

where $D_1, \ldots, D_k$ are division algebras over our base field K. For $i = 1, \ldots, k$ let Z_i be the center of D_i. Z_i is a field, as you do not need to show. Show that A is separable if and only if the field extension $K \subset Z_i$ is separable for all $i = 1, \ldots, k$. (6 points)

Problem 3: Let H be a Hopf algebra. Show that the counit is not distinguished among all the algebra homomorphisms to the base field in the following sense: If $\gamma : H \to K$ is another algebra homomorphism, show that there is a new comultiplication and a new antipode for H so that H, with the same algebra structure, the new comultiplication, γ as a counit, and the new antipode is again a Hopf algebra.
(Hint: What are the properties of the map $f : H \to H,\ h \mapsto \gamma(h_{(1)})h_{(2)}$?)
(4 points)

Problem 4: Let $H := \mathbb{C}[\mathbb{Z}]$ be the group algebra of the integers over the complex numbers, written as finite Laurent polynomials in the variable z. Define

$$\sigma : \mathbb{C}[\mathbb{Z}] \to \mathbb{C}[\mathbb{Z}], \quad \sum_{i=-\infty}^{\infty} a_i z^i \mapsto \sum_{i=-\infty}^{\infty} \bar{a}_i z^{-i}$$

where $\bar{a}$ denotes the complex conjugate of a.

1. Show that the fixed points H^σ of σ form a Hopf subalgebra of H over $\mathbb{R}$.
2. For a subgroup $n\mathbb{Z} \subset \mathbb{Z}$, show that the group algebra $A := \mathbb{C}[n\mathbb{Z}]$ and its fixed points A^σ are Hopf subalgebras over $\mathbb{C}$ and $\mathbb{R}$, respectively.
3. Show that H^σ is not free over A^σ if n is even. (6 points)

Due date: Tuesday, April 1, 2014. Please write your solution on letter-sized paper, and write your name on your solution. It is not necessary to submit this sheet with your solution.