University at Buffalo
Yorck Sommerhäuser

Spring Semester 2014
MTH 819: Sheet 7

Hopf Algebras

Problem 1: If A is an algebra, an element $c = \sum_{i=1}^{m} x_i \otimes y_i$ is called a Casimir element if

$$\sum_{i=1}^{m} a x_i \otimes y_i = \sum_{i=1}^{m} x_i \otimes y_i a$$

for all $a \in A$. The Casimir element is called symmetric if

$$\sum_{i=1}^{m} x_i \otimes y_i = \sum_{i=1}^{m} y_i \otimes x_i$$

1. Show that the set of Casimir elements and the set of symmetric Casimir elements are subspaces of $A \otimes A$. (1 point)

2. If $A = M(n \times n, K)$ is the algebra of $n \times n$-matrices, show that

$$c = \sum_{i,j=1}^{n} E_{ij} \otimes E_{ji}$$

   is a symmetric Casimir element, where E_{ij} is the (i,j)-th matrix unit. Show that every symmetric Casimir element is a scalar multiple of c. (3 points)

3. Find all Casimir elements in the algebra $A = M(n \times n, K)$, and determine the dimension of the space of Casimir elements. (4 points)

Problem 2: A Casimir element $c = \sum_{i=1}^{m} x_i \otimes y_i$ is called a separability idempotent if $\sum_{i=1}^{m} x_i y_i = 1$. An algebra with separability idempotent is called separable. (It is called strongly separable if the separability idempotent is symmetric.) Show that a separable algebra is semisimple. (Hint: Try to imitate the proof of Maschke's theorem.) (4 points)

Problem 3:

1. Show that a semisimple Hopf algebra is separable. (2 points)

2. Show that a semisimple Hopf algebra over a field of characteristic zero is strongly separable. (2 points)

Problem 4: Show that a separable algebra is finite-dimensional. Conclude that a semisimple Hopf algebra is finite-dimensional. (Hint: Be careful not to get into circular arguments.) (4 points)

Due date: Tuesday, March 25, 2014. Please write your solution on letter-sized paper, and write your name on your solution. It is not necessary to submit this sheet with your solution.