**University at Buffalo**
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**Spring Semester 2014**
**MTH 819: Sheet 4**

# Hopf Algebras

**Problem 1:** Suppose that $A$ is a bialgebra, and that $V$ and $W$ are left comodules over $A$.

1. Show that $V \otimes W$ is again a comodule when endowed with the coaction
$$\delta_{V\otimes W}(v \otimes w) := v^{(1)}w^{(1)} \otimes v^{(2)} \otimes w^{(2)}$$
2. If $U$ is another left comodule over $A$, show that the canonical isomorphism of vector spaces
$$(V \otimes W) \otimes U \cong V \otimes (W \otimes U)$$
is colinear.
3. Show that the base field $K$ becomes a comodule over $A$ when endowed with the coaction
$$\delta_K : K \to A \otimes K,\ \lambda \mapsto 1 \otimes \lambda$$
4. Show that the canonical isomorphisms $K \otimes V \to V,\ \lambda \otimes v \mapsto \lambda v$ and $V \otimes K \to V,\ v \otimes \lambda \mapsto \lambda v$ are colinear. (4 points)

**Problem 2:** Suppose that $H$ is a Hopf algebra, and that $V$ is a finite-dimensional left comodule over $H$ with basis $v_1, \ldots, v_n$ and dual basis $v_1^*, \ldots, v_n^*$.

1. Show that the dual space is again a left comodule over $H$ when endowed with the coaction
$$\delta_{V^*} : V^* \to H \otimes V^*,\ \varphi \mapsto \sum_{i=1}^{n} S(v_i^{(1)})\varphi(v_i^{(2)}) \otimes v_i^*$$
2. Show that this coaction satisfies
$$\varphi^{(1)}\varphi^{(2)}(v) = S(v^{(1)})\varphi(v^{(2)})$$
for all $\varphi \in V^*$ and all $v \in V$.
3. Show that the evaluation map
$$\mathrm{ev} : V \otimes V^* \to K,\ v \otimes \varphi \mapsto \varphi(v)$$
is colinear. (3 points)

**Problem 3:** Let $L$ be a Lie algebra. An associative algebra $U$ (with unit) together with a linear map $\iota : L \to U$ is called a universal enveloping algebra if

$$\iota([x,y]) = \iota(x)\iota(y) - \iota(y)\iota(x)$$

for all $x, y \in L$ and the following universal property holds: If $A$ is another associative algebra (with unit) together with a linear map $i : L \to A$ satisfying

$$i([x,y]) = i(x)i(y) - i(y)i(x)$$

for all $x, y \in L$, then there is a unique (unit-preserving) algebra homomorphism $f : U \to A$ satisfying $f \circ \iota = i$. Show that there is a unique Hopf algebra structure on $U$ with the following properties:

1. $\Delta(\iota(x)) = \iota(x) \otimes 1 + 1 \otimes \iota(x)$ for all $x \in L$.
2. $\varepsilon(\iota(x)) = 0$ for all $x \in L$.
3. $S(\iota(x)) = -\iota(x)$ for all $x \in L$.

(Remark: It is a consequence of the (nontrivial) Poincaré-Birkhoff-Witt theorem that $\iota$ is injective. This fact is not needed for this problem.) (6 points)

**Problem 4:** Decide whether the Taft algebras $T$ defined on Exercise Sheet 2 are semisimple. If semisimplicity depends on properties of the parameter $q$, find and state these properties. Prove all your assertions in complete detail. (7 points)

Due date: Tuesday, February 25, 2014. Please write your solution on letter-sized paper, and write your name on your solution. It is not necessary to submit this sheet with your solution.